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BACKGAMMON

How To Maximize Your Profits And Optimize Your Cube Decisions Using The Kelly System

BY: MICHELIN CHABOT

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BACKGAMMON, HOW TO MAXIMIZE YOUR PROFITS AND OPTIMIZE YOUR CUBE DECISIONS USING THE KELLY SYSTEM

TABLE OF CONTENTS

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I also express my gratitude to well-known expert Kathy Posner who kindly accepted to write the foreword.

I would also like to thank all my friends who read the manuscript, especially Réginald Proulx and Narvey Goldman, and my wife, Lise, for her constant encouragement.

Michelin Chabot

FOREWORD

In recent years, there has been a plethora of backgammon books that deal with checker movement, cube strategy, match play and game theory. These books are important tools to build the foundation of a player. Michelin Chabot has taken us past the ground level. His first book, Backgammon: How Much Should You Bet? (reviewed by me in Backgammon Times) introduced us to the concept of money management and the variables to be considered for maximizing our monetary success.

I was very impressed with his theories and put them into practice with my own betting. They were very beneficial. I felt more in control of my gambling because I was more clearly aware of how to achieve my goal of winning. I was more confident with my cube decisions because I was applying sound principles of money management.

With the application of the Kelly System, Michelin has now taken us to the second floor of the backgammon house. I was astounded by what I learned on this level. I have been playing and gambling for thirteen years and thought I knew what was going on. The Incredible Chabot Paradoxical Proposition was a revelation to me. I had always assumed that if I continuously played a non-contact position, with any size bet, in which I was a favorite, that I would end up a winner. **THAT IS NOT TRUE !**

When you read this book, and study it well you should, without a doubt your game strategy has to improve. The formulas are fairly easy to calculate before you sit down to play so there is no excuse for not establishing your optimal bet. Of course if your opponent has also read the book his bet decision may be different from yours !

There is no reason not to master the principles of money management in this book. Michelin has created a masterpiece of theory and formulas which should become "the book" of backgammon gambling.

Kathy Posner

Kathy Posner was nominated for backgammon's "Giant 32" and has been playing for 13 years. She won numerous tournaments, the latest being the 1983 Chicago Cup in February. She is a contributions editor for Backgammon Times and loves chocolate chip cookies.

INTRODUCTION

To make money playing backgammon, you need two elements: 1) skill advantage and 2) good money management principles, which mainly means: correctly establishing your bet and handling the cube properly.

In my first book: Backgammon: How Much Should You Bet?, I explained different objectives you may have and how to establish your appropriate bet according to your goal.

This book should be considered as a complement to my first one, and its objective is to introduce an additional goal. When you have a skill advantage (i.e. you are involved in a favorable game), you may have as a goal to maximize the rate of increase of your wealth. John L. Kelly Jr. worked with this goal and established a theory known as the Kelly theory. This theory permits us to derive our optimal bet and to optimize our cube decisions.

When my first book was published in May 1982, I knew the Kelly theory was applicable to strategy games in general. Indeed, it has been used for years by blackjack players and sport bettors. However, I was under the impression that it could not be used in backgammon because, in practice, it is almost impossible to vary the bet from one game to an other. Since then, I found out that in spite of this difficulty and others, it is indeed both possible and profitable to use the Kelly system.

Chapter 1 explains the Kelly theory. The second chapter elaborates how this system should be used for games in general. In Chapter 3, we will consider what adjustments should be made to establish the optimal bet for backgammon. In Chapter 4, we will see the consequences of the Kelly criterion on the doubling cube theory. In Chapter 5, we will expose the dilemma generated by the Kelly theory. Finally, in Chapter 6, the ten commandments of money management will be given.

The main characteristic of the Kelly theory is that the bets are expressed in **percentage of bankroll** rather than in dollars. To understand why, consider the following example. You are betting $100 on an event. Are you betting too high or too little? Obviously, the question cannot be answered unless you compare this amount to your bankroll. That is why it is often desirable to think in percentage terms rather than in dollars.

Since most of us are not used to think in percentage terms, we might have some temporary difficulties to adjust, in the same way that we experience difficulties adjusting to computers. However, once we do get used to it, we realize it is more practical and reach the point where we no longer use the old system at all.

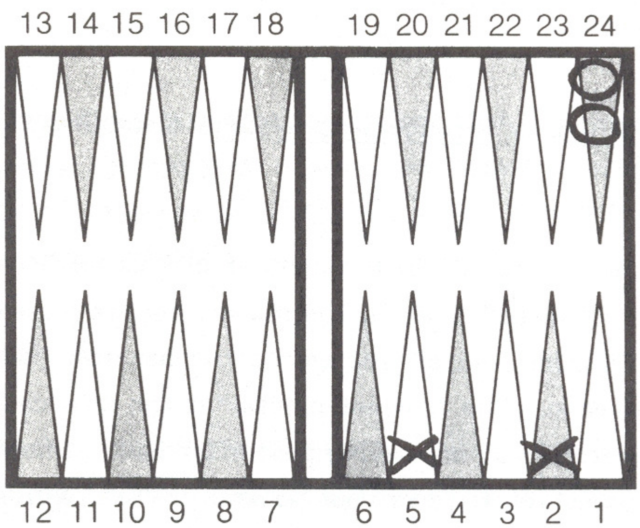
A word of advice: do not be surprised if, after the first reading of this book, you find you do not understand many concepts or examples. This is perfectly normal and I recommend you give it a second reading, step by step, followed by as many readings as needed to thoroughly understand it. You may find it tedious, but I am confident you will quickly be rewarded in material gains if you apply the Kelly theory correctly.

* * * * *

CHAPTER 1: THE KELLY MONEY MANAGEMENT SYSTEM

In 1956, John L. Kelly Jr. published an article entitled: New Interpretation of Information Rate. Kelly explained that if you always bet 100% of your capital in a very favorable game in which your chances of winning are less than 100%, you will eventually be broke. In this sense, one might say that Kelly mathematically confirmed the old saying: "Do not put all your eggs in the same basket". He also demonstrated that there exists an optimal bet which maximizes the rate of increase of your wealth.

Let's suppose you play the following backgammon position as a proposition:



X to play
No cube

You probably know that **X** has a probability of winning of 19/36. If your initial capital is $1,000 and you bet 100% of your cumulative capital on each trial and play against a very rich opponent, you will eventually be broke and your opponent will make money with you ! Between the extreme cases of betting 0% and 100% of your capital, there is an optimal percentage which will maximize the rate of increase of your wealth. The Kelly theory states that if you use that optimal percentage, 1) on the long run you are 100% certain to make more money than using any other percentage, and 2) your chance of going broke is theoretically 0%.

The theory associated with the Kelly system is relatively difficult to understand for a "non-mathematical reader", but surprisingly, the Kelly criterion is very easy to use. Because of the complexity of the theory, the formulas will not be developed, but rather be given in their simplest form. Many practical examples will explain how to proceed. The reader interested in the mathematical implications of the formulas explained in this chapter should consult the following writings:

- Kelly, John L. Jr., A New Interpretation of Information Rate, Bell System Technical J. 35, No. 4 (July 1956), pp. 917-926.

- Epstein Richard A., The Theory of Gambling and Statistical Logic (1977), Academic Press Inc., pp. 60-62.

- Wilson, Allan W., The Casino Gambler's Guide, Harper & Row Publishers (Enlarged Edition), pp. 298-300.

- Thorp Edward O., The Kelly Money Management System, Gambling Times, December 1979, pp. 91-92.

Before going on, let us define the term "capital" in the context of this publication. Your capital is the equivalent of your bankroll, that is, the money you are willing to risk (for gambling purposes). The "initial capital" represents the bankroll you are willing to risk at the start of the session. The "current capital" is the new capital, taking into account winnings and losses. If you are paying cash after each game, your capital represents the amount of money you have in your pockets and are willing to risk.

If a gambler bets a fraction "F" of his current capital at each trial (or play), his exact future capital will be evaluated as follows:

$$\text{Exact Future Capital} = (1 + F)^W \times (1 - F)^L \times \text{Initial Capital}$$

F = Fraction of the current capital bet on each trial

W = Number of wins

L = Number of losses

Formula 1

The above formula is extracted from Kelly's article.

Example 1:

Your initial capital is $1,000. You bet 10% of your current capital on five consecutive games (the payoff is even). What will be your exact future capital if you win 3 times?

Using Formula 1:

$$\text{Exact Future Capital} = (1 + F)^W \times (1 - F)^L \times \text{Initial Capital}$$

with F = 10%, W = 3, L = 2, Initial Capital = $1,000, we obtain:

$$\text{Exact Future Capital} = (1 + .1)^3 \times (1 - .1)^2 \times \$1{,}000 = \$1{,}078.11$$

* * * * *

You should pay attention to the fact that the results obtained using Formula 1 do not depend on the sequence of wins and losses. Therefore, if we represent a win by W and a loss by L, the sequence W W W L L or L L W W W or L W W W L will give exactly the same result: $1,078.11.

Example 2:

Given the following assumptions: Initial Capital = $1,000, F = 10%, W = 3, and L = 2, could you demonstrate that the sequence W W W L L will give exactly the same results as sequence L L W W W (or as any other similar sequence)?

With the sequence W W W L L, the calculations proceed as follows:

Bets ($)	Results W: Win L: Loss	Current Capital ($)
100.00	W	1,000.00 + 100.00 = 1,100.00
110.00	W	1,100.00 + 110.00 = 1,210.00
121.00	W	1,210.00 + 121.00 = 1,331.00
133.10	L	1,331.00 - 133.10 = 1,197.90
119.79	L	1,197.90 - 119.79 = 1,078.11

With sequence L L W W W, we have:

Bets ($)	Results W: Win L: Loss	Current Capital ($)
100.00	L	1,000.00 - 100.00 = 900.00
90.00	L	900.00 - 90.00 = 810.00
81.00	W	810.00 + 81.00 = 891.00
89.10	W	891.00 + 89.10 = 980.10
98.01	W	980.10 + 98.01 = 1,078.11

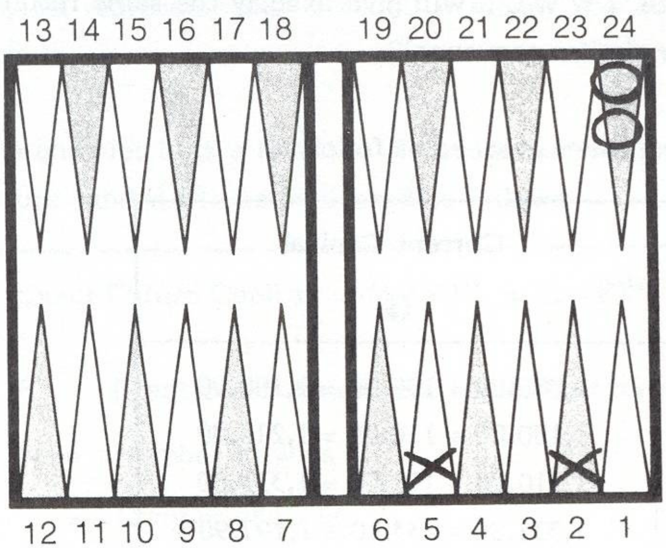
Any other sequence will give the same final result: $1,078.11.

* * * * *

Let us now use Formula 1 to solve a more complicated problem.

Example 3:

A player sets up the following backgammon position:



X to play
No cube

and presents you with the following proposition:

1) you play as **X**;
2) your initial capital is $1,000;
3) you bet a fixed fraction of your current capital on each trial;
4) you perform 36 trials.

Since you know your probability of winning each game is 19/36, you accept the proposition. You decide to bet 20% of your current capital. What will be your future bankroll if you obtain 19 sucessful throws? What would your future capital have been had you decided to use 5%?

Using Formula 1 with F = 20%, W = 19, L = 17, Initial Capital = $1,000, we obtain:

$$\text{Exact Future Capital} = (1.2)^{19} \times (.8)^{17} \times \$1{,}000 = \$719.41$$

Using Formula 1 with F = 5%, we obtain:

$$\text{Exact Future Capital} = (1.05)^{19} \times (.95)^{17} \times \$1{,}000 = \$1{,}056.57$$

The results may seem paradoxical but they are rigourously exact. They do not prove that you will lose money if you bet 20% of your current capital and win if you bet 5%; they are valid, if and only if, you obtain 19 successful throws.

* * * * *

It is often interesting to know the average ratio by which your capital will be multiplied on each trial. The following formula permits us to calculate that ratio (it is derived from the previous formula).

$$R = (1 + F)^P \times (1 - F)^Q$$

R = Average Ratio by which the capital will be multiplied after each trial

F = Fixed Fraction of your current capital

P = Probability of winning

Q = Probability of losing (Q = 1-P)

Formula 2

This formula is extracted from the book: The Casino Gambler's Guide, pp. 299.

It should be noted that:

1) When R is higher than 1.0, your capital will grow.
2) When R is lower than 1.0, your capital will decrease.
3) When R equals 1.0, your capital will remain constant.

The diagram "R Vs F" has the following shape:

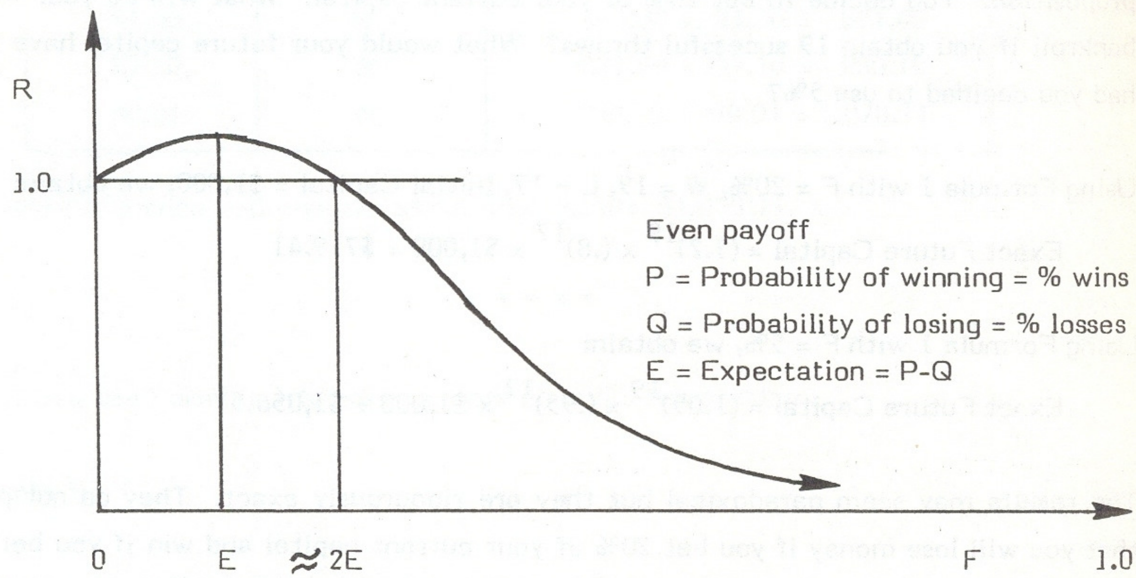


Diagram 1:
R Versus F
R = Average Ratio by which the capital will multiplied after each trial
F = Fixed Fraction of your current capital

This diagram shows that:

1) The optimal fraction F is equal to the expectation E = P-Q.
2) When F is lower than about 2 x E, your capital will grow.
3) When F is higher than about 2 x E, your capital will decrease in the long run.

Basing ourselves on the previous graph, when the payoff is even, the Kelly criterion permits us to calculate the optimal bet as follows:

Optimal bet = (% wins - % losses) x Current Capital	Formula 3

The readers of my first book will notice that the expression: (% wins - % losses), is equivalent to the expectation "E" when the payoff is even. It should be noted that when you obtain a negative optimal bet, the game is unfavorable and should generally not be played. For the exception to this, see Section 6.3.

Your expected capital is established as follows:

Expected Capital = R^N x Initial Capital		
R =	Average Ratio by which your capital is multiplied after each trial (see Formula 2)	
N =	Number of trials	Formula 4

Example 4:

Your initial capital is $1,000. You are involved in a game in which your probability of winning is 19/36. You bet 5% of your current capital at each trial. What is the average ratio by which your capital will be multiplied after each step? What should your bankroll be after 360 trials?

The value of R is obtained using Formula 2:

$$R = (1 + F)^P \times (1 - F)^Q$$

In this case, we have F = 5%, P = 19/36 and Q = 17/36, so:

$$R = (1.05)^{19/36} \times (.95)^{17/36} = 1.00153$$

After 360 trials, the expected capital can be evaluted using Formula 4:

$$\text{Expected Capital} = R^N \times \text{Initial Capital}$$

$$\text{Expected Capital} = 1.00153^{360} \times \$1{,}000$$

$$\text{Expected Capital} = \$1{,}734.$$

* * * * *

In order to build a diagram in which we will replace "R" by the "expected bankroll", let us suppose that we are involved in a favorable game in which our probability of winning is 19/36. The payoff is even. Let us assume we will play 360 games. We can calculate the expected capital with different "F" values ranging from 0% to 15% with 1/2% steps. Using calculations similar to those in the preceeding example, we derive the following table:

Table 1

Value of your expected capital in function of "F" (the fraction bet)

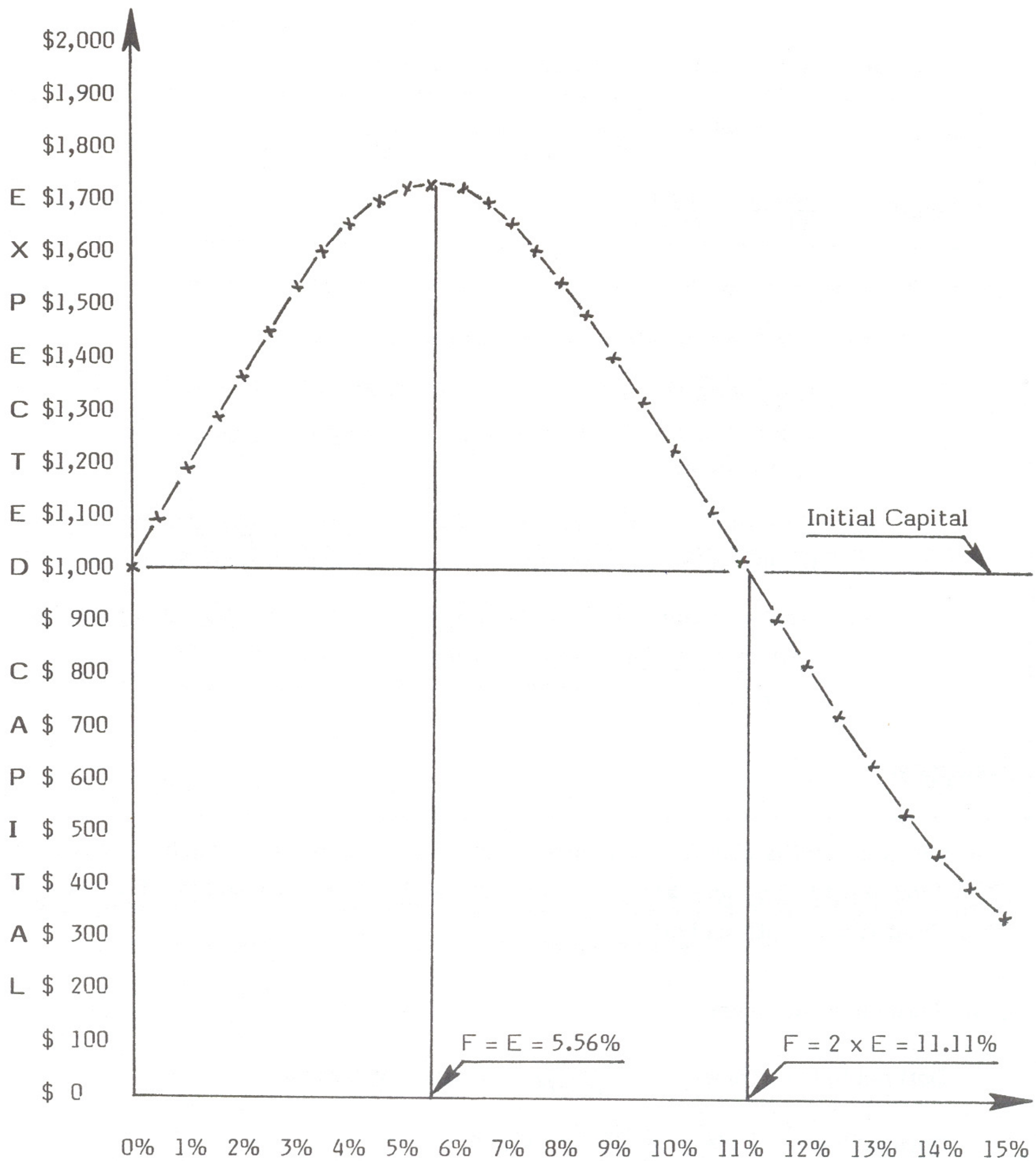
F	Expected Capital ($)	F	Expected Capital ($)
0%	1,000	8.0%	1,565
0.5%	1,100	8.5%	1,490
1.0%	1,200	9.0%	1,406
1.5%	1,296	9.5%	1,315
2.0%	1,388	10.0%	1,219
2.5%	1,473	10.5%	1,119
3.0%	1,550	11.0%	1,018
3.5%	1,616	11.5%	917
4.0%	1,669	12.0%	819
4.5%	1,709	12.5%	725
5.0%	1,734	13.0%	636
5.5%	1,743	13.5%	552
6.0%	1,737	14.0%	475
6.5%	1,716	14.5%	405
7.0%	1,679	15.0%	342
7.5%	1,628		

In a graph form:

Diagram 2:

Expected Capital Vs "F"

(% Wins = 19/36, N = 360, Initital Capital = $1,000)



F = Fraction of the current capital bet on each trial

Diagram 2 clearly shows that we obtain the optimal percentage at point F equalling E = P-Q = 19/36 - 17/36 = 2/36 = 5.56%. It also shows that beyond point F = 2E, the expected capital is inferior to the initial one (i.e. you will lose money).

* * * * *

When the payoff is not even, the Kelly criterion becomes:

$$\text{Optimal bet} = (\text{\% wins} - \frac{\text{\% losses}}{\text{payoff}}) \times \text{Current Capital}$$

$$\text{Payoff} = \frac{\text{Amount you can win}}{\text{Amount you can lose}}$$

Formula 5

Notes: 1) When the payoff is even, Formula 5 is equivalent to Formula 3.

2) Formula 5 is equivalent to the ones presented by:

a) Huey Mahl in the Frontline, issue no. 23, September 7-13, 1979, pages 1 and 8 and in his technical paper "Money Management" (December 12-13, 1979).

b) Edward O. Thorp's article: The Kelly Money Management System, published in the Gambling Times, December 1979, pp. 91-92.

c) Ronald Roblin & John Slivka in the article: Applying the Kelly Criterion at The Track, published in the Gambling Times, February 1983, pp. 10-14.

Example 5:

Your initial capital is $1,000. You wish to bet on a sport event in which the payoff is 10/11 (you win $100 or lose $110). You estimate that your probability of winning is 60%. What is your optimal bet?

Using Formula 5, we have:

$$\text{Optimal bet} = (\text{\% wins} - \frac{\text{\% losses}}{\text{payoff}}) \times \text{Current Capital}$$

$$\text{Optimal bet} = (60\% - \frac{40\%}{10/11}) \times \$1{,}000$$

$$\text{Optimal bet} = 16\% \times \$1{,}000 = \$160$$

Example 6:

Your capital is $1,000. You wish to bet on a horse race. Your horse pays 5 to 2. You estimate that this horse has a winning probability of 30%. What is your optimal bet?

Using Formula 5, we have:

$$\text{Optimal bet} = (\%\text{ wins} - \frac{\%\text{ losses}}{\text{payoff}}) \times \text{Current Capital}$$

$$\text{Optimal bet} = (30\% - \frac{70\%}{5/2}) \times \$1{,}000$$

$$\text{Optimal bet} = 2\% \times \$1{,}000 = \$20$$

* * * * *

When the payoff is uneven, Formula 2 can be modified as follows to take the payoff into account:

$$R = (1 + A \times F)^{P} \times (1 - F)^{Q}$$

R = average Ratio by which your capital will be multiplied after each trial

$$A = \text{Payoff} = \frac{\text{Amount you can win}}{\text{Amount you can lose}}$$

F = Fixed Fraction of your current capital

P = Probability of winning

Q = Probability of losing (Q = 1-P)

Formula 6

Note: When the payoff is uneven, the expected bankroll is still obtained by using Formula 4, but "R" should be derived from Formula 6.

Example 7:

Your present capital is $1,000. You are involved in a game in which your probability of winning is 4% and the payoff is 35 to 1. What fraction of your bankroll should you bet? What is the average ratio by which your capital will be multiplied after each trial? What is your expected capital after 1,000 games?

Your optimal bet is obtained by using Formula 5:

$$\text{Optimal bet} = (\%\text{ wins} - \frac{\%\text{ losses}}{\text{payoff}}) \times \text{Current Capital}$$

$$\text{Optimal bet} = (4\% - \frac{96\%}{35}) \times \text{Current Capital}$$

$$\text{Optimal bet} = 1.257\% \times \text{Current Capital}$$

The value of "R" (the average ratio by which your capital will be multiplied after each trial) is obtained using Formula 6:

$$R = (1 + A \times F)^P \times (1 - F)^Q$$

$$R = (1 + 35 \times .01257)^{0.04} \times (1 - .01257)^{0.96}$$

$$R = 1.00244$$

After 1,000 trials, your expected capital is evaluted using Formula 4 as follows:

$$\text{Expected Capital} = R^N \times \text{Initial Capital}$$

$$\text{Expected Capital} = 1.00244^{1000} \times \$1,000$$

$$\text{Expected Capital} = \$11,439$$

Note: This example uses the assumptions given by Edward O. Thorp in his previously cited article. The results are similar even if the approach is different.

* * * * *

To conclude this chapter, I would like to explain why the Kelly system is applicable to backgammon. Many people seem to believe that the Kelly system cannot be applied to backgammon, altough they admit it is useful in blackjack or sports betting. Because of this, I would like to clear up their confusion. The fact is the Kelly system is applicable to **any** game where the player's expectation is positive. Backgammon certainly fits in this category, along with blackjack and sports betting, whereas games like roulette or craps do not.

Another argument people use is that because the bet cannot be changed from game to game, the Kelly theory is unusable. I agree that in most cases, the bet stays constant for a series of games, but that does not mean that the Kelly theory is useless. On the contrary, I think the best approach is to calculate your optimal bet at the start of the games (see Chapter 3) and use it for the whole session. The rate of increase of your bankroll will not be as high as if you would vary the bet from one game to the other, but it will most probably be higher than if you used guesswork in establishing your bet.

Finally, other people think that because of the presence of the cube, the Kelly theory is not valid. I disagree. The cube is taken into account both in establishing the unit bet, by using the cube factor, and in the other aereas as well. Therefore, I strongly believe that using the Kelly system in backgammon is possible and profitable. I further believe that no serious player can afford to ignore it.

In summary, the Kelly Money Management System merely consists of using Formula 3 if the payoff is even and Formula 5 if the payoff is uneven. The main advantages of using the Kelly criterion are:

1) The probability of ruin is practically "Nil" because you always bet a fixed fraction of your capital.

2) You maximize the rate of increase of your capital.

3) You minimize the time needed to obtain your desired bankroll.

4) You will make more money faster than using flat betting or any progressive "system".

* * * * *

CHAPTER 2: THE KELLY SYSTEM APPLIED TO GAMES IN GENERAL

2.1 The winning probability is constant

In this case, your optimal bet is established using Formula 5. Your only concern is to evaluate your winning probability. Since often in practice, your bet has to be a multiple of $1, $2, $5, or $10, you have to round off the theoretical optimal bet to find your practical optimal bet.

Example 8:

You are involved in a game in which your winning probability is 19/36. The payoff is even. Your initial capital is $1,000. The amount bet has to be a multiple of $5. What is the practical optimal bet?

Since your probability of winning is 19/36, your probability of losing is 17/36, and your expectation is calculated as follows: E = (probability of winning - probability of losing). In our example, E = 19/36 - 17/36 = 2/36. Therefore, in theory, you should bet 2/36 of your current capital in each trial. Your theoretical first bet of 2/36 x $1,000 = $55.56 should be rounded off to $55. If you win, your current capital will become $1,055 and your second bet should be 2/36 x $1,055 = $58.61, which can be rounded off to $60, and so on.

* * * * *

When you are involved in a game in which it is possible to calculate your winning probability, all you have to do is use Formula 5 and round off the result.

2.2 The winning probability is variable

When the payoff is even and your probability of winning cannot be exactly established, you have to apply the following formula:

$\text{Expectation} = \dfrac{\text{Cumulative Gains (or losses)}}{\text{Cumulative Wagered Amount}}$	Formula 7

Example 9:

You play a game of chance and skill in which you either win \$10 or lose \$10 (no ties). After 100 trials, you have obtained 55 wins. What is your expectation? What will your expectation be after this game?

You have won 55 games at \$10/game and lost 45 games at \$10/game. Your cumulative gains are \$100. Your cumulative wagered amount is 100 games x \$10/game = \$1,000. Your expectation is therefore $\frac{\$100}{\$1{,}000} = 10\%$. After this game, your expectation will be different according to your losses or wins.

Case 1: You win

Your cumulative gains become \$110 and the cumulative wagered amount is \$1,010. Your expectation becomes $\frac{\$110}{\$1{,}010} = 10.9\%$.

Case 2: You lose

Your cumulative gains become \$90, so your expectation is now $\frac{\$90}{\$1{,}010} = 8.9\%$.

* * * * *

When the expectation is variable, the optimal bet can be evaluated as follows:

$$\text{Optimal bet} = \frac{\text{Cumulative Gains}}{\text{Cumulative Wagered Amount}} \times \text{Current Capital} \qquad \text{Formula 8}$$

Example 10:

You have played 100 games of chance and skill. The bet was \$10 at each trial and you succeeded in making cumulative gains of \$100. Your present capital is \$200. How should you proceed, in theory? What should be your first bet? What should be your next bet?

In theory, you should constantly calculate your cumulative gains, your cumulative wagered amount and your current capital and finally establish your bet using Formula 8.

Your cumulative gains are \$100, your cumulative wagered amount is 100 games x \$10/game = \$1,000, your optimal bet is:

$$\frac{\$\,100}{\$1{,}000} \times \$200 = \$20$$

Your next bet should be different depending on whether you win or lose.

Case 1: You win

Your cumulative gains become \$120, your cumulative wagered amount is now \$1,020 and your current capital becomes \$220. Based on Formula 8, your optimal bet is $\frac{\$\,120}{\$1{,}020} \times \$220 = \$25.88$ which can be rounded off to \$25.

Case 2: You lose

Your cumulative gains become \$80, your cumulative wagered amount is now \$1,020 and your current capital becomes \$180. The optimal bet is $\frac{\$\,80}{\$1{,}020} \times \$180 = \$14.11$ which can be rounded off to \$15.

* * * * *

CHAPTER 3: ADJUSTMENTS FOR BACKGAMMON

3.1 Calculation of your expectation

In the context of this chapter, the term "expectation" means: your expectation while playing against a specific opponent. This concept has nothing to do with the concept of expectation related to a backgammon position. The first kind of expectation could be labelled as the "player's expectation" and the latter as the "expectation of a backgammon position". In my book: Backgammon: How Much Should You Bet? (published in May 1982), I explain how to evaluate the "player's expectation". This expectation depends on your skill and on your opponent's (this subject will be further developed in my next book entitled: Backgammon: Luck, Skill or Art?). If you sharpen your skill while your opponent's remains constant, it implies that your "player's expectation" will increase. At the moment, the only practical way to evaluate your (player's) expectation is to rely on previously achieved results. Another approach would be to use a rating system (I also develop this point in my future book), but there is no such system at present.

In order to estimate your expectation, Formula 7 must be used:

$$\text{Expectation} = \frac{\text{Cumulative Gains (or losses)}}{\text{Cumulative Wagered Amount}}$$

To estimate your cumulative wagered amount, you could use the following formula:

$$\text{Cumulative Wagered Amount} = \text{Amount bet per point} \times C_f \times N$$

$$C_f = \text{Cube factor} = \frac{\text{Number of points}}{\text{Number of games}} = \frac{\text{True average bet}}{\text{Basic bet}}$$

N = Number of games

Formula 9

In my book, Backgammon: How much should you bet?, I give an example on how to estimate your (player's) expectation for the game of backgammon. I will repeat it, very slightly modified.

Example 11:

You played backgammon at $5 a point and obtained the following results:

Game number	Result of each game	Cumulative results
1	+1	+1
2	-2	-1
3	+4	+3
4	-1	+2
5	+2	+4
6	-1	+3
7	+4	+7
8	-2	+5
9	+1	+6
10	-2	+4

A) What were your cumulative gains?

You won four points or $20.

B) What was the cumulative wagered amount?

The cumulative amount wagered was established by using Formula 9. In this example, we have $5 bet per point. The number of games is 10. The cube factor is established by adding all points wagered (whether won or lost) divided by the number of games. Your cube factor is established as follows:

- Number of games (won or lost) ending with 1 point (4 games x 1 point): 4 points
- Number of games (won or lost) ending with 2 points (4 games x 2 points): 8 points
- Number of games (won or lost) ending with 4 points (2 games x 4 points): 8 points

Total: 20 points

$$C_f = \text{Cube factor} = \frac{\text{Number of points}}{\text{Number of games}} = \frac{20 \text{ points}}{10 \text{ games}} = 2.0 \text{ (points per game)}$$

Therefore, using Formula 9, your cumulative wagered amount is:

$$\text{Cumulative wagered amount} = \text{Amount bet per point} \times C_f \times N$$

$$\text{Cumulative wagered amount} = \frac{\$5}{\text{point}} \times \frac{2.0 \text{ points}}{\text{game}} \times 10 \text{ games} = \$100$$

Your cumulative wagered amount is \$100, with 10 games played. Therefore, the average amount wagered per game is \$10. For this example, your bet is **\$5 a point** and because your cube factor is 2.0 (points/game) you are playing, in fact, for **\$10 a game.**

C) What is your (player's) expectation?

Your expectation is calculated using Formula 7:

$$\text{Expectation} = \frac{\text{Cumulative Gains (or losses)}}{\text{Cumulative Wagered Amount}} = \frac{\$\,20}{\$100} = 20\%$$

* * * * *

After the publication of my book (Backgammon - How Much Should You Bet?), Danny Kleinman wrote an article entitled "Using Results to Estimate Expectations", in which he disagrees with my way of establishing the expectation. You will find his article in Appendix A and my reply in Appendix B.

Kleinman's conclusion is:

> "In short, "average" expectations are meaningless to the extent that they mask individual difference stemming from significant variables."

Since expectation is based strictly on past results, and this concept is the "heart" of the Kelly criterion which permits us to derive our **optimal bet,** I do not share Kleinman's conclusion. I am rather convinced that this concept is among the most important ones to be considered in backgammon since expectation is the best tool to select your opponents and establish your optimal bet.

In conclusion, to establish your current (player's) expectation, it is necessary that you keep track of your results, calculate your cumulative gains, your cumulative wagered amount (using Formula 9) and your new expectation after each session. However, in calculating your expectation, you should have two concerns:

1) Discard results you feel are unrepresentative.

2) Evaluate the reliability of your expectation related to the number of trials performed. The more trials (games played) you performed, the more accurate is your expectation. For example, if the expectation obtained after 10 games is 20%, it would be hazardous to be willing to bet 20% of your bankroll, but if this 20% is obtained after 1,000 games, you could consider it accurate.

Since the reliability of the expectation against an opponent is more reliable as the number of "representative" games you played against him increases, we suggest the use of a "safety factor", which depends on the number of games played. It has been mathematically established that you need at least 30 games to start calculating the expectation, and that when you have played, say 1,000 games, you are almost certain the expectation you have calculated is accurate. It is therefore suggested to multiply the expectation by a "safety factor" labelled K_1 using the following table:

Table 2

Safety Factor (K_1)

Number of representative games played against your opponent	Suggested value of K_1
30 to 100	0.5
100 to 250	0.7
250 to 500	0.8
500 to 1000	0.9
1000 and up	1.0

3.2 Corrections to be made when the bet is increased after a specific number of games

In a sport event, you can bet $200 on a first event and adjust the next bet to $250 or $150 according to your result on the first bet.

A blackjack counter could bet $5 when the deck is new and 1% or 2% of his capital if, later on, the deck is favorable to him. If his capital is $10,000, it means that, in a very short time, he can increase his bet from $5 to $100 and, later on, reduce it to $5 at the beginning of the next shuffle.

In backgammon, the bet is constant for a certain number of games. And most of the time, when the bet is changed, it is toward the end of the session and is often increased. Because of this restriction we have to make some adjustments. Considering the number of games to be played before the bet can be changed, it is necessary to adjust your optimal bet. This adjustment should lower your optimal bet. The suggested adjustment is:

$K_2 = \frac{R^N}{C}; K_2 \leq 1.0$

K_2 = Coefficient number 2

R = Average ratio by which the capital will be multiplied after each step, as defined in Formula 2

N = Number of games to be played before the bet can be increased

C = Ratio by which the stakes are multiplied after N games

$\leq$ = Inferior or equal to

Formula 10

The reader wishing to know the mathematical derivation of Formula 10 is referred to Appendix C where K_2 is discussed more completely. It should be noted that K_2 takes a maximum value of 1.0. In other words, if you obtain a "K_2" value greater than 1 in your computations, use $K_2 = 1$. Computation of C is easy; it is equal to the bet after the stakes are increased, divided by the bet at the beginning of the session. For example, if you raise the bet from \$20 to \$50 a point, then $C = \frac{\$50}{\$20} = 2.5$. The reader should bear in mind that when the bet remains constant during the session, then K_2 takes a value of 1.0.

Example 12:

You regularly play against the same opponent, and you usually double the stakes after a specified number of games. You have determined from past results, that your probability of winning against him is 60%. What is the "K_2" value when the number of games to be played before the bet is doubled is successively: 10, 20 and 50?

Your optimal fraction of bankroll is:

F = % wins - % losses = 60% - 40% = 20%

According to Formula 2:

$$R = (1 + F)^P \times (1 - F)^Q = (1.2)^{0.6} \times (0.8)^{0.4} = 1.02034$$

and we have: $C = 2$.

Using Formula 10, we obtain the following "K_2" values:

N	K_2
10	.61
20	.75
50	1.0

It should be noted that for N = 50, the actual value obtained for K_2 is 1.37 and since it is greater than 1.0, we use $K_2 = 1.0$.

* * * * *

3.3 Establish your optimal bet for matches

You usually play matches against the same opponent. You have kept track of the results and now wish to know if your bet is the optimal one. In matches, your optimal bet is established as follows:

Optimal bet for matches = E x Present Capital x K_1 x K_2

$$E = \text{Expectation} = \frac{\text{Cumulative Gains}}{\text{Cumulative Wagered Amount}}$$

K_1 = Safety factor established using Table 2

K_2 = Coefficient calculated using Formula 10

Formula 11

Example 13:

In a regular session, you play five 9-point matches. You have always bet $50 a match, but now your opponent insists on doubling the stakes on the last match. So far, you have played 80 matches and won 45. Your present capital is $3,000. What is your optimal bet?

Since the number of matches played is 80, according to Table 2, the safety factor "K_1" is equal to .5.

Your cumulative gains are 10 matches x \$50/match = \$500. Your cumulative wagered amount is 80 matches x \$50/match = \$4,000. Your expectation is \$500/\$4,000 = 12.5%. The minimum number of matches to play without changing the bet is 4. Your optimal rate of increase is:

$$R = (1 + E)^P \times (1 - E)^Q$$

$$= (1.125)^{45/80} \times (0.875)^{35/80} = 1.00786$$

The value of K_2 is therefore:

$$K_2 = \frac{R^N}{C} = \frac{(1.00786)^4}{2} = 0.52$$

Your optimal bet is calculated using Formula 11:

Optimal bet for matches = E x Present Capital x K_1 x K_2

Optimal bet for matches = 12.5% x \$3,000 x .5 x .52 = \$97.50

Your practical optimal bet is \$100 a match.

* * * * *

In the preceeding example, your usual bet of \$50 was too low. If you are afraid to raise the stakes directly from \$50 to \$100, you should increase it by steps. If your goal is to maximize the rate of increase of your capital, you should adher to the Kelly criterion and bet the optimal way.

3.4 Establish your optimal bet for money games

In money games, you have to consider the cube. The cube factor (C_f) represents the relation between the average amount bet per game and the amount bet per point. In money games, your optimal bet is obtained as follows:

$$\text{Optimal bet for money games} = \frac{E \times \text{Present Capital} \times K_1 \times K_2 \times K_3}{C_f}$$

$$E = \text{Expectation} = \frac{\text{Cumulative Gains}}{\text{Cumulative Wagered Amount}}$$

The "Cumulative Wagered Amount" is established using Formula 9:

K_1 = Safety factor established using Table 2

K_2 = Coefficient calculated using Formula 10

K_3 = Coefficient related to the dilemma explained in Chapter 5. Unless specified, $K_3 = .9$

C_f = Cube factor

Formula 12

Example 14:

After having played 1,000 games against the same opponent at an average of $5 a point and a cube factor of 2.0 (points per game), you won $500. Your present capital is $1,000. In a regular session, you play about 50 games and you can double the bet after 30 games. What is your expectation against this player? What is your optimal bet?

Your cumulative gains are $500. Your cumulative wagered amount is established using Formula 9:

$$\text{Cumulative wagered amount} = \text{Amount bet per point} \times C_f \times N$$

$$\text{Cumulative wagered amount} = \frac{\$5}{\text{point}} \times \frac{2 \text{ points}}{\text{game}} \times 1{,}000 \text{ games}$$

$$\text{Cumulative wagered amount} = \$10{,}000.$$

Your expectation is established as follows:

$$E = \text{Expectation} = \frac{\text{Cumulative Gains}}{\text{Cumulative Wagered Amount}} = \frac{\$500}{\$10{,}000} = 5\%$$

Since the number of games played is 1,000, Table 2 give us: $K_1 = 1.0$. The number of games to be played before the bet can be changed is 30; the coefficient K_2 is therefore .52. Your optimal bet is established using Formula 12:

$$\text{Optimal bet for money games} = \frac{E \times \text{Present Capital} \times K_1 \times K_2 \times K_3}{C_f}$$

$$\text{Optimal bet for money games} = \frac{5\% \times \$1{,}000 \times 1.0 \times .52 \times .9}{2.0} = \$11.70 \text{ (a point)}$$

Your practical optimal bet is $10 a point. Your usual bet of $5 is too low.

* * * * *

The Kelly criterion suggests that, when playing money games, you should always try to wager your optimal bet. If your optimal bet is $10 a point, you should try to convince your opponent who wishes to play at $5 a point to increase the bet. The criterion also implies that you should never be willing to bet well beyond your optimal bet.

When you are involved in a chouette, you could use Formula 12 to establish your optimal bet. "E", the expectation achieved, should be the expectation you had playing against this specific chouette and "C_f", the cube factor, should also be evaluated taking into account your results in this specific chouette.

Example 15:

Each Friday night you play in the same four-man chouette. So far, you have played 1,000 games, the bet was $5 a point and your cube factor was 5.0; meaning that your average bet is $25 a game. Your cumulative gains are $1,000. Your present capital is $5,000. You usually play 50 games and the bet is never raised. What is your optimal bet?

Your cumulative wagered amount is 1,000 games $\times \frac{\$25}{\text{game}} = \$25{,}000$. Since your cumulative gains are $1,000, your expectation is $1,000 / $25,000 = 4%. Coefficients K_1 and K_2 take the value 1.0. Coefficient K_3 is equal to .9; your optimal bet is then:

$$\text{Optimal bet for money games} = \frac{E \times \text{Present Capital} \times K_1 \times K_2 \times K_3}{C_f}$$

$$\text{Optimal bet for money games} = \frac{4\% \times \$5{,}000 \times 1.0 \times 1.0 \times .9}{5.0} = \$36$$

Your practical optimal bet is $35 a point. You should try to convince your "chouette" to increase the bet of $5 a point which is obviously too low.

* * * * *

3.5 Establish your "optimal entrance fee" for tournaments

Risking money in playing matches, money games or in tournaments is basically the same thing. You have to consider the prizes given, the expenses involved and your probability of winning. Formula 5 should also be used to find your optimal bet. In this section, no additional formula is given and the approach to use will be explained by a few very simple examples.

Example 16:

You are invited to participate in a tournament. The entrance fee is $200. 20% of these fees are kept for organizational expenses. Thirty-two players will be present. There is only one prize established as follows: $200 x 32 x 80% = $5,120. Since this tournament is held in your city, there will be no additional expenses for transportation, food or room. You evaluate your winning probability at around 6%. Your actual gambling capital is $1,000. Your goal is to maximize the rate of increase of your capital. Should you enter this tournament?

To solve this problem, we can first estimate your optimal bet (i.e. entrance fee). We could use Formula 5:

$$\text{Optimal bet} = (\%\text{ wins} - \frac{\%\text{ losses}}{\text{payoff}}) \times \text{Current Capital}$$

$$\text{Payoff} = \frac{\text{Amount you can win}}{\text{Amount you can lose}} = \frac{\$5,120}{\$200} = 25.6$$

% wins = probability of winning = 6%

Current capital = $1,000

Optimal bet = (6% - 94%/25.6) x $1,000

Optimal bet = 2.33% x $1,000 = $23.30.

The entrance fee is too high since your "optimal entrance fee" is $25. To be willing to risk $200 when the optimal bet is $25 is an over-bet. Since the Kelly theory discourages "overbetting" you should not enter this tournament. If your capital was $8,000, it would be an optimal investment.

* * * * *

Example 17:

You enter a tournament. After having paid the entrance fee, your gambling capital is \$5,000. The total pool for the auction is \$10,000 and this amount will be given to the buyer of the champion. You have been bought for \$200. You evaluate your probability of winning at around 3% (based on past results in tournaments with opponents of the same caliber). You have the possibility to buy yourself back up to 50% (i.e. 50% of \$200). Your goal is to maximize the rate of increase of your capital. What is the optimal proportion of yourself you should buy back?

Using Formula 5 we have:

$$\text{Optimal bet} = (\%\text{ wins} - \frac{\%\text{ losses}}{\text{payoff}}) \times \text{Current Capital}$$

$$\text{Payoff} = \frac{\$10{,}000}{\$200} = 50.0$$

$$\%\text{ wins} = 3\%$$

$$\text{Optimal bet} = (3\% - 97\%/50.0) \times \$5{,}000$$

$$\text{Optimal bet} = 1.06\% \times \$5{,}000 = \$53$$

Even if you are allowed to buy back 50% of yourself, your optimal proportion to buy back is 25% which represents \$50.

* * * * *

3.6 How to handle "Double or Nothing" propositions

It often happens that, near the end of a session in which you are ahead, your opponent will offer you to play a "Double or Nothing" game, which means a game without the cube and gammons. That game is played for the amount of money you won during the session.

When facing such a situation, the Kelly system once again enters the picture. The approach is very simple: if the amount you play for in this game is well under your optimal bet, accept the offer, if and only if you are sure there is no payment problem. If, however, the amount is near or well above your optimal bet, refuse to play "Double or Nothing". Here is an example:

Example 18:

Your gambling bankroll is $10,000 and you have determined your advantage is 5%, based on a large number of games. Near the end of the session, you are $50 ahead and you know your opponent will offer you several "Double or Nothing" games in a row. There is no payment problem. Your goal is to maximize the rate of increase of your capital. Should you accept to play "Double or Nothing"? How many such games should you play?

Since you have a $10,000 bankroll and a 5% advantage, your optimal bet is $500. Therefore, you should accept "Double or Nothing" games for $50, $100, $200 and $400 but do not go beyond that.

* * * * *

CHAPTER 4: EFFECT OF THE KELLY CRITERION ON THE DOUBLING CUBE THEORY

The Kelly criterion (i.e. Formula 5) is applicable to all games in which the player has an advantage. Most of the time, the player's advantage is very difficult to establish with precision; for example, consider sport events or horse racing. In backgammon, because you can double the stakes during the game, it is somewhat different; you have a specific position and you can often calculate your exact probability of winning. If your advantage is 10%, you should be willing to bet 10% of your current capital, so, if the game is played for 1% of your capital and you have an opportunity to double, do so. However, if the game is played for 15% of your capital, you should refrain from doubling unless you are certain your opponent will drop or accept a correct settlement. **Because in backgammon you can often establish your probability of winning, the Kelly criterion is a very useful tool in the doubling cube theory.**

4.1 The Incredible Chabot Paradoxical Proposition

If you study Diagram 1, you will observe that in the long run, you will lose money in a **favorable** game if your bet is too high. Diagram 1 slightly modified is reproduced hereunder:

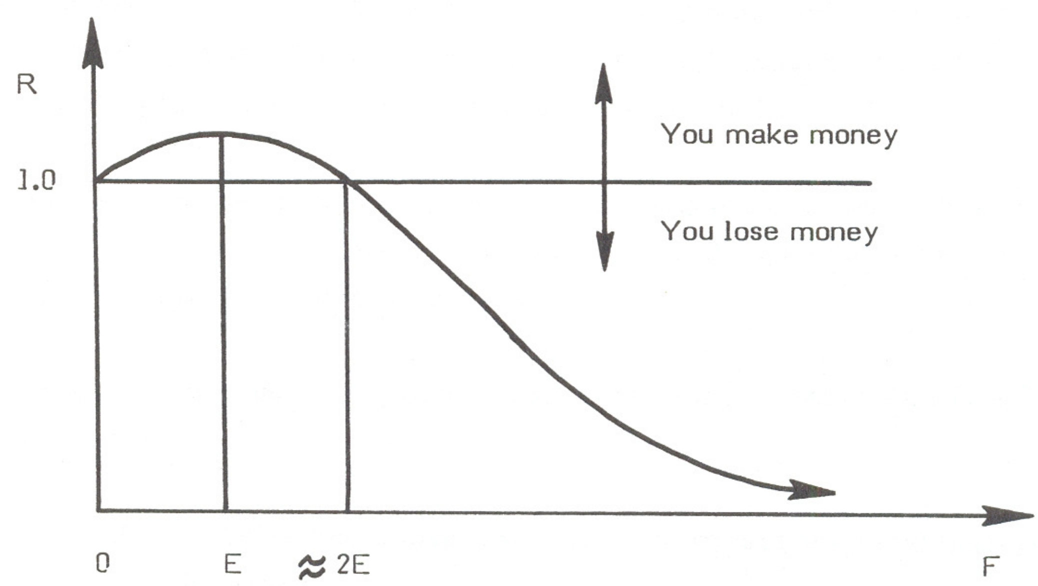


Diagram 3:

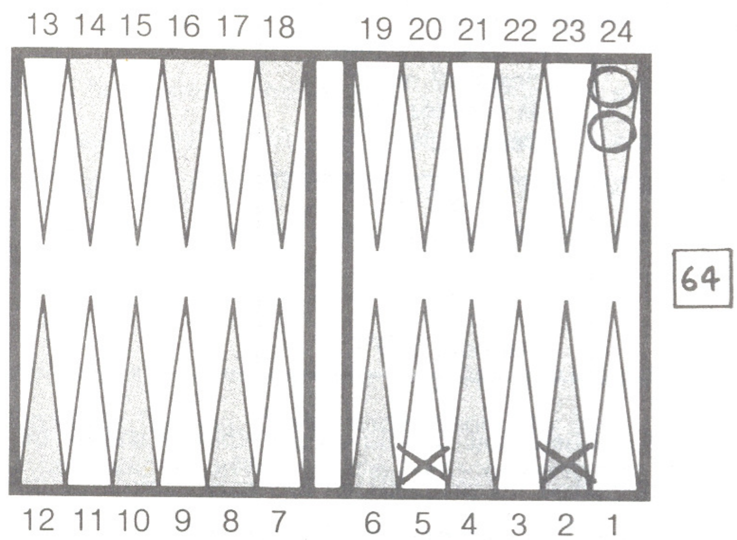
R Versus F

R = Average Ratio by which your capital will be multiplied after each trial

F = Fixed Fraction of your current capital

The fact that you can lose money in a favorable game seems to be a paradox; but such a thing happens if the fraction you bet exceeds twice your expectation. Taking this consideration into account, you could make money in an unfavorable game if your opponent overbets. Once again, the preceeding statement appears to be incredible. Now with this in mind, let's take a look at the following proposition.

THE INCREDIBLE CHABOT PARADOXICAL PROPOSITION



X to play

X's bet = 5% of his current capital

Initial capital = $1,000

Cube is centered

If X doubles, O can beaver

This position will be played as a proposition as follows:

1) **You play as X. You always double and I (as O) always beaver.** In each play, you bet 5% of your current capital. For example, your first bet is 5% x $1,000 = $50; you double and I beaver; if you win, your current capital becomes $1,200 and your second bet is 5% of $1,200 i.e. $60. The sequence of wins and losses along with your current capital are kept on an appropriate recording sheet. After 360 trials (or plays), your current capital is labelled "capital no. 1".

2) **I play as X and never double.** I use the exact sequence you have obtained in your 360 trials. I then keep track of my current capital as we did with yours and the capital obtained is labelled "capital no. 2".

If your capital (capital no. 1) is greater than my capital (capital no.2), I lose $1,000; if not, I win $1,000.

Note: 1) There is no payment after each trial; the results are merely compiled on a recording sheet.

2) The difference between "capital no. 1" and "capital no. 2" is not paid by either player; these capitals are only "paper capitals".

* * * * *

Appendix D gives a complete analysis of my proposition. In summary, the probability that "capital no. 2" is greater than "capital no. 1" is related to the number of trials.

Number of trials	**Probability that "capital no. 2" is greater than "capital no. 1"**
36	69.1%
360	90.7%
1,000	98.6%
10,000	almost 100%

The goal of my proposition is to clearly establish that doubling cube theory should be related to your current capital. I have suggested my proposition to many backgammon players, but so far, I have had no takers. Here is a summary of their reactions:

1) Indeed, it looks illogical, but I do not accept because you are known as a mathematician and you probably made all the calculations. If you suggest a proposition, it must be to your advantage.

2) I will play this position any time with a fixed (and uniform) bet only. For example $10 a point for 1,000 games. I am not interested in betting a percentage of my current capital because, in practice, it doesn't work like that.

3) Even if you could prove by this proposition that betting 20% of your current bankroll is worse than betting only 5%, such a conclusion would not have any consequences on a short-term basis.

Notes: a) The concept of expectation is a long-term one which has consequences on a short-term basis. The concept of rate of increase of your capital is also a long-term one which, obviously, has also a short-term application.

b) Most of the real "bad luck stories" in backgammon happen when a player gives the cube at a very high level to maximize his expectation while, according to the Kelly criterion and his current capital, such a cube should not be given.

c) Even if you argue that one single cube decision will not have a noticeable effect on your final results, it is the cumulative result of wrong cube decisions that make an important impact on your long-term capital.

4) I have always believed that a redouble at level 4 was not necessarily a redouble at level 8 or 16 but I was not able to express it mathematically in a way that would take into consideration the very important point of the capital I can afford to risk. So, I agree with your proposition but I cannot demonstrate why it is to your advantage.

My proposition seems very illogical and almost unbelievable but it is strictly based on the Kelly theory. If you understand why my proposition is a winning one, you will also understand the effect of the Kelly criterion on the doubling cube theory.

If the expectation of a backgammon position (for the last roll) is 2/36 = 5.56%, your optimal bet is 5.56% of your current capital. To be willing to bet 20% of your capital on a position in which your expectation is around 5% is a losing proposition. The maximum percentage of your bankroll you should be willing to bet should never exceed twice the expectation of the position reached. **The doubling cube theory is related to your current capital.**

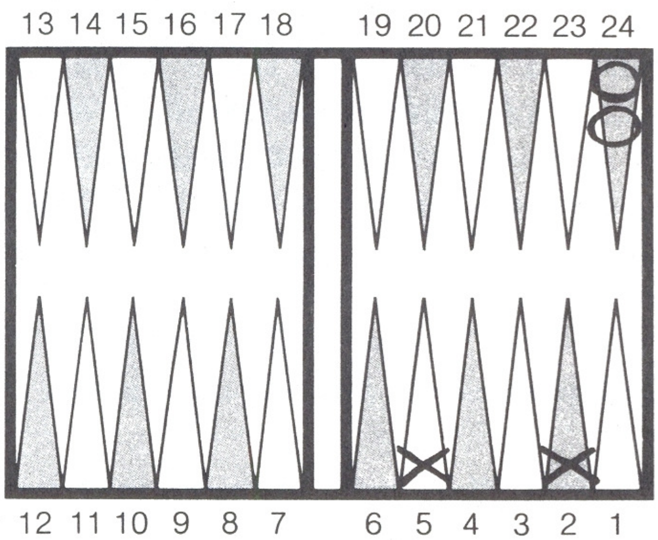
4.2 Criteria to determine if you should double

Many well known theoreticians have stated that backgammon players have only one goal. For example, in his book: The Doubling Cube in Backgammon (Volume I), Jeff Ward states: "A player always wants to maximize his equity, so he should only double when doubling furthers this goal" (page 36). Also, Paul Magriel in his book Backgammon, states: "From a theoretical viewpoint, the question of accepting doubles (and offering them also) should be considered independently of the level of the cube. In other words, it is never theoretically correct to accept a double at the 8 level which you would pass at the 2 level or vice versa" (page 274).[1] Bill Robertie also made a similar statement in his book: Lee Genud vs Joe Dwek, The 1981 World Championship of Backgammon.

Any theory is a coherent ensemble of the three following elements: 1) assumption(s), 2) goal(s) and 3) solution(s). All are inter-related. If one assumption changes, then the solution may change. If the goal changes, the solution may also change. The solution given must satisfy all assumptions and goals. The doubling cube theory obeys the same rules.

So, for a specific backgammon problem, the player having as a goal to maximize his expectation will not necessarily reach the same answer as the player who wishes to maximize the rate of increase of his capital. Here is an example:

Example 19:



Beavers are not allowed

Bet = $10 a point

Should **X** double or redouble whatever the level of the cube?

A) **X**'s goal is to maximize his expectation.

B) **X**'s goal is to maximize the rate of increase of his capital of $400, $800, $1,600 and $3,200.

A) Since **X**'s goal is to maximize his expectation, he should double and redouble whatever the level of the cube. The above position is a classic one.

B) Now **X**'s goal is different, so the solution should change.

If X's bankroll is $400, his optimal bet is 2/36 x $400 = $22.22. X should try to reach this optimal bet. Since the bet is $10 a point, X should give the cube at level 2 but if he has the cube at level 2, he should keep it because at this level he has already reached his optimal bet for this position.

If X's capital is $800, his optimal bet is 2/36 x $800 = $44.44. He should double to level 2, redouble to level 4 but not redouble beyond this level.

If X's bankroll is $1,600, his optimal bet is 2/36 x $1,600 = $88.89. Since the bet is $10 a point, the optimal level for the cube is 8. X should try to reach this level and not go beyond. If his capital were $3,200, the optimal cube level would be 16.

In summary, if X's goal is to maximize the rate of increase of his capital, we have:

Should X give the cube?

Level of the cube (after being offered)	**X's capital is:**			
	$400	**$800**	**$1,600**	**$3,200**
2	yes	yes	yes	yes
4	no	yes	yes	yes
8	no	no	yes	yes
16	no	no	no	yes
32	no	no	no	no

* * * * *

If your goal is to maximize the rate of increase of your capital, you should use the following formula to determine if you should double:

If	$R_d > R_{nd} \Longrightarrow$	you should double
R_d	=	Average Ratio by which your capital will be multiplied after each trial if you double
$>$	=	Greater than
R_{nd}	=	Average Ratio by which your capital will be multiplied after each trial if you do not double
$\Longrightarrow$	=	It implies that
R_d	=	$(1 + A \times F_d)^P \times (1 - F_d)^Q$
A	=	Payoff $= \dfrac{\text{Amount you can win}}{\text{Amount you can lose}}$
F_d	=	Fraction of your capital wagered if you double
P	=	Probability of winning
Q	=	Probability of losing (Q = 1-P)
R_{nd}	=	$(1 + A \times F_{nd})^P \times (1 - F_{nd})^Q$
F_{nd}	=	Fraction of your capital wagered if you do not double

Formula 13

Example 20:

For the preceeding example, in which your probability of winning is 19/36 and in which the bet is $10 a point, could you prove, using a mathematical approach, that if your capital is $1,600, you should give the cube at level 8 and refrain from giving it at level 16 and 32?

You should use Formula 13. In this case, the payoff is even, so A = 1. The fraction bet is related to the cube level. When the cube is at level 4, the fraction of your capital bet is 2.5%. The calculations are as follows:

Should you give the cube from... to...	Your fraction bet will vary from... to...	Rate of increase if you do not double (R_{nd})	Rate of increase if you double (R_d)	Correct cube decision
4 to 8	2.5% to 5%	1.00108	1.00153	redouble
8 to 16	5% to 10%	1.00153	1.00055	keep the cube
16 to 32	10% to 20%	1.00055	0.99089	keep the cube

You should give the cube at level 8 because R is greater at level 8 than at level 4 but you should refrain from giving it beyond level 8.

* * * * *

Obviously, when you are playing, you cannot make those calculations. The best approach is simply to calculate the expectation of your position (in %) and multiply this result by your current capital. You then obtain your optimal bet.

The following graph shows the influence of the Kelly theory on the doubling strategy in one-shot positions, with no gammon possibilities and no beavers:

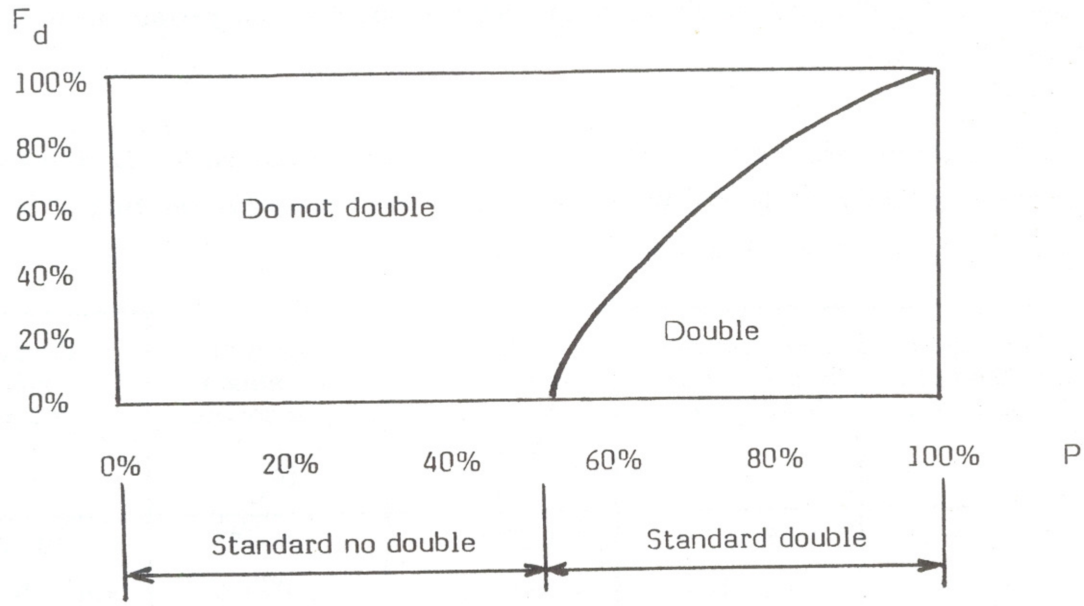


Diagram 4: Doubling strategy for one-shot positions (with no gammons)

F_d = Fraction of your capital wagered after you double

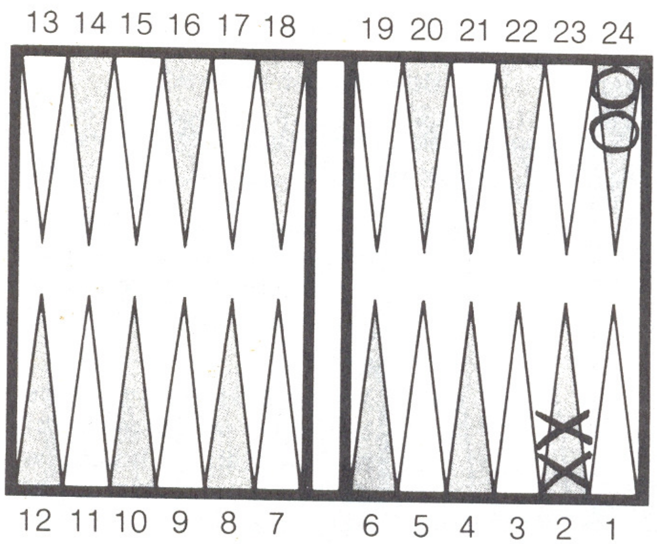
P = Your probability of winning

The mathematical derivation of Diagram 4 is given in Appendix E.

Diagram 4 clearly shows that the standard doubling strategy is a particular case of the Kelly strategy. When the fraction bet is low, the standard theory is valid. The higher this fraction becomes, the more the Kelly strategy differs from the standard one.

Example 21:

You have reached the following position:



Beavers are not allowed

X to play

X's capital = $1,000

Bet = $10 a point

Should **X** give the cube at any level?

There are two practical approaches: the first one consists of calculating your optimal bet, and the second, more precise, uses Diagram 4.

Approach no. 1:

The probability of winning is 26/36. X's expectation is 16/36. X's optimal bet is 16/36 x $1,000 = $444.44. Since the bet is $10 a point, the optimal cube level is 32. **X** should give the cube up to level 32 but should refrain from giving it at the 64-level.

Approach no. 2:

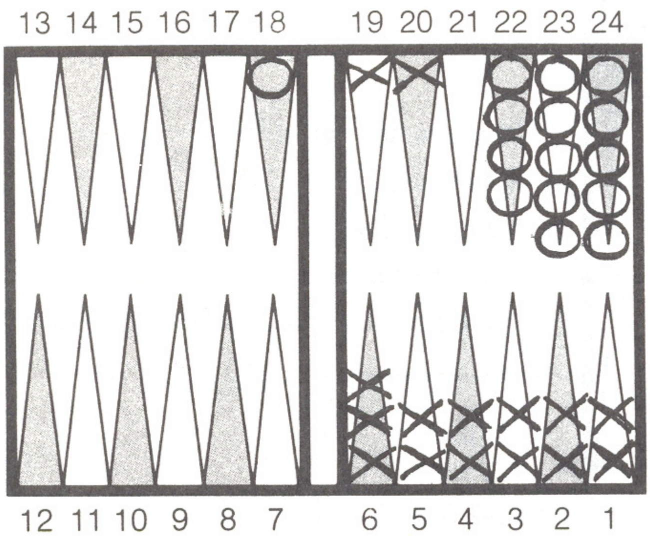
Referring to Diagram 4, it can be seen that **X** should double if F_d is around 58% or less. This means that **X** should double up to level 32.

* * * * *

It should be noted that: 1) when you know your opponent will drop a double, you are justified to double and 2) if you are convinced your opponent will accept a correct settlement, you are also justified to double (I intend to develop this subject in a future book).

Example 22:

(Inspired from: The Doubling Cube in Backgammon (Volume I), by Jeff Ward, page 129).



Beavers not allowed

X's bankroll = $1,000

X to play

A) X's goal is to maximize his expectation. Should he double or redouble whatever the level of the cube?

B) X's goal is to maximize the rate of increase of his capital. Should he double or redouble whatever the level of the cube with unit bets of $1, $5, $10 and $50?

A) Since X is favored to hit (20/36 56%) and this roll will determine the outcome of the game, the player wishing to maximize his expectation should double or redouble whatever the level of the cube. By so acting, X doubles his expectation. If X does not give the cube, he simply loses his market.

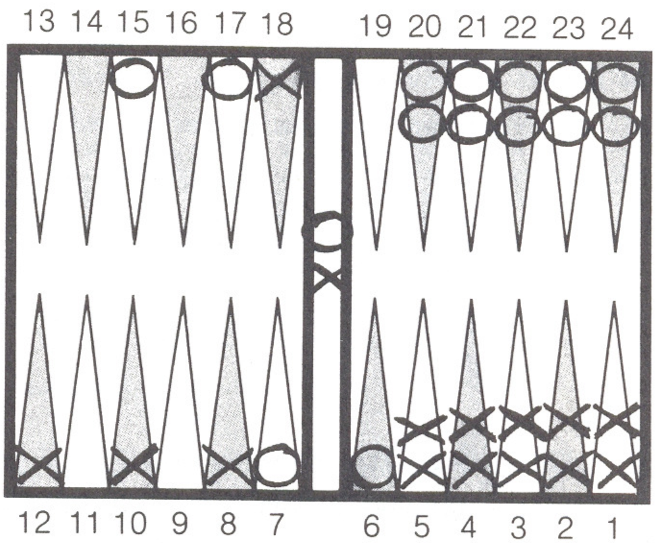
B) X's goal is now different. So, the solution to this problem is different. Since X's probability of winning is 20/36, his probability of losing is 16/36 and his advantage is 4/36 = 1/9. X's optimal bet is thus 1/9 of his gambling capital. Since X's capital is $1,000, the optimal bet is 1/9 x $1,000 = $111.11. If the bet is $1 a point, X should be willing to raise the cube up to level 128 and refrain from redoubling beyond this level. If the bet is $5 a point, X should be willing to reach level 16 and not redouble beyond this level. If the bet is $10 a point, X should be willing to redouble up to level 8. If the bet is $50 a point, X should double to level 2 but never redouble higher.

* * * * *

Example 23:

(Inspired from "Riding the Tiger" by Bob Floyd, published in the Las Vegas Backgammon Magazine, April/May 1982, pp. 34-35).

You have reached the following position:



Beavers not allowed

X to play

Bet = $10 a point

X's capital = $2,000

X has the cube

You may remember that Bob Floyd's analysis demonstrates that 1) the player on roll will win this game with a probability of 36/61 (59%), 2) the result will be a gammon whether he wins or loses, and that 3) this position could be considered as a "perpetual redouble" which means that as long as the position doesn't change, the player on roll should redouble. Your goal is to maximize the rate of increase of your capital. Should you (as **X**) give the cube at any level? What is the optimal cube level?

Since your probability of winning the game is 36/61 (59%), your probability of losing is 25/31 (41%) and your expectation is 11/61 = 18%. Your optimal bet is 18% x $2,000 = $360. Since the bet is $10 a point, and the result is necessarily a gammon, the optimal cube level is 16. You should be willing to raise the cube until it reaches this level and refrain from going beyond it.

Note: 1) If the bet were $5 a point, the optimal cube level would be 32.

2) We can conclude that, despite Bob Floyd considering this position as a "perpetual redouble", it is not so if we accept the fact that one or both players do not have "unlimited bankrolls".

* * * * *

Exemple 24: The eye of the Tiger !

(Inspired from "The tail of the Tiger", by Danny Kleinman in his book: Double-Sixes from the Bar, page 81-82).

You are involved in a game in which there are no men, only two dice and the cube. You are **X** and own the cube at level 2. If you roll double-6's, you win, if not, it is your opponent's turn to play. Your opponent will have the opportunity to give you the cube if he owns it. The first player who rolls double-6's will win the game. You remember that 1) Kleinman's analysis (correctly) demonstrates that the player on roll has a probability of winning of 36/71 (50.7%) 2) he also considers his proposition as a "perpetual redouble" 3) he suggests the appointment of a "trusted banker" to make sure the players can afford the stakes indicated by the cube level. Your gambling capital is $10,000 and the bet is $1 a point. Your goal is to maximize the rate of increase of your capital. Should you give the cube at any level?

Since your probability of winning is 36/71, your probability of losing is 35/71 and your expectation is 1/71. Your optimal bet is 1/71 x $10,000 = $141. The optimal cube level is 128. You should be willing to redouble from 2 to 4, accept the cube to level 8, redouble at level 16, accept the cube at level 32, redouble to the 64-level, accept the cube at level 128 but **you should refrain from giving the cube at level 256.**

Notes: 1) I feel that even if Kleinman correctly stresses that "The only rational money management aim is to maximize money expectation over the long run" ("Double sixes from the Bar", page 114), he does not know how to guide his readers on how to proceed.

2) I agree with Kleinman that the proposition is a "perpetual redouble" only if the two players are infinitely rich and want to maximize their expectation at each trial. However, I disagree with his suggestion that each player should double until one of them risks 100% of his bankroll. Both players should not go beyond 1/71 of their bankrolls if they wish to maximize the rate of increase of their wealth.

3) If you correctly use the Kelly criterion, you will be in contact with the banker for deposits !

* * * * *

When you reach a position in which you either win a gammon with a probability P or lose a simple game, your optimal bet is obtained using Formula 5:

$$\text{Optimal bet} = \left(\%\text{ wins} - \frac{\%\text{ losses}}{\text{payoff}}\right) \times \text{Current Capital}$$

$$\text{Payoff} = \frac{\text{Amount you can win}}{\text{Amount you can lose}}$$

Formula 5

Example 25:

You have reached a one-shot position in which you either win a gammon with a probability of 15/36 or lose a simple game. The bet is $10 a point. Your capital is $2,500. What is your optimal bet? Should you give the cube at any level?

Your optimal bet is evaluated using Formula 5 in which % wins = 15/36 and Payoff = 2. So you have:

$$\text{Optimal bet} = \left(\%\text{ wins} - \frac{\%\text{ losses}}{\text{payoff}}\right) \times \text{Current Capital}$$

$$\text{Optimal bet} = (15/36 - (21/36)/2) \times \$2{,}500 = 12.5\% \times \$2{,}500 = \$312.50$$

Since the bet is $10 a point, the optimal level of the cube is 32. You should give the cube up to the 32-level and refrain from giving it beyond that point.

* * * * *

4.3 Criteria to determine if you should drop, take or beaver

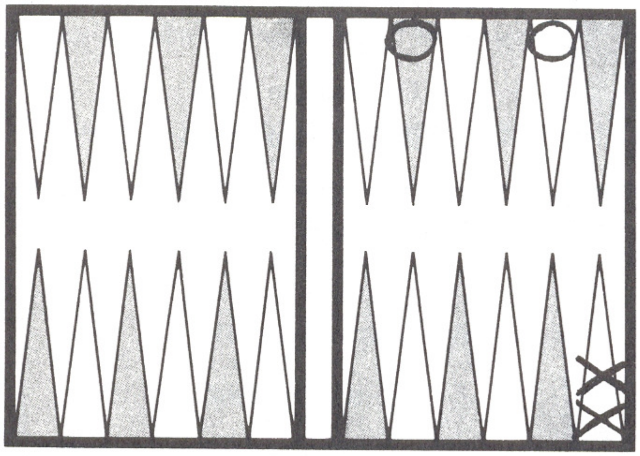
In the previous section, it was shown that the "standard" doubling behavior (i.e. double when you have the advantage) is incomplete, since it does not take into account the level of the cube or the player's bankrolls. Similarly, in this section, it will be seen that the well-known 3 to 1 accepting rule is also incomplete for the same reasons.

If your goal is to maximize the rate of increase of your capital or, when the position is unfavorable, to minimize the rate of decrease of your capital (the two expressions are equivalent), to decide if you should accept the cube, you should use the following formula:

If	$R_a > R_r \Longrightarrow$	you should accept
R_a	=	Average Ratio by which your capital will be multiplied after each trial if you accept
$>$	=	Greater than
R_r	=	Average Ratio by which your capital will be multiplied after each trial if you refuse
$\Longrightarrow$	=	It implies that
R_a	=	$(1 + A \times F_a)^P \times (1 - F_a)^Q$
A	=	Payoff $= \dfrac{\text{Amount you can win}}{\text{Amount you can lose}}$
F_a	=	Fraction of your capital wagered if you accept
P	=	Probability of winning
Q	=	Probability of losing ($Q = 1-P$)
R_r	=	$(1 + A \times F_r)^{0\%} \times (1 - F_r)^{100\%} = 1 - F_r$
F_r	=	Fraction of your capital wagered if you refuse

Formula 14

Example 26:



O gives the cube

Bet = $10 a point

X's capital = $1,000

Should **X** accept the cube at any level?

For **X**, we have: P = 17/36 and Q = 19/36. If **X** drops at level 2, we have:

$$R_r = 1 - F_r = 1 - 1\% = .99$$

and if **X** accepts at level 2, we have:

$$R_a = (1 + A \times F_a)^P \times (1 - F_a)^Q$$

$$R_a = (1 + 1 \times 2\%)^{17/36} \times (1 - 2\%)^{19/36} = .99869$$

Since $R_a > R_r \Rightarrow$ **X** should accept. For other levels we have:

Level of the cube after being offered by O	Fraction bet if accepted F_a	$R_a = (1+A \times F_a)^P \times (1-F_a)^Q$ (A=1, P=17/36, Q=19/36)	$R_r = 1-F_r$ ($F_r = .5\,F_a$)	X's cube decision
2	2%	0.99869	.99	Take
4	4%	0.99698	.98	Take
8	8%	0.99237	.96	Take
16	16%	0.97831	.92	Take
32	32%	0.93012	.84	Take
64	64%	0.73668	.68	Take
128	128%	0.0	.36	Drop

You should take up to level 64. You cannot afford to take at level 128.

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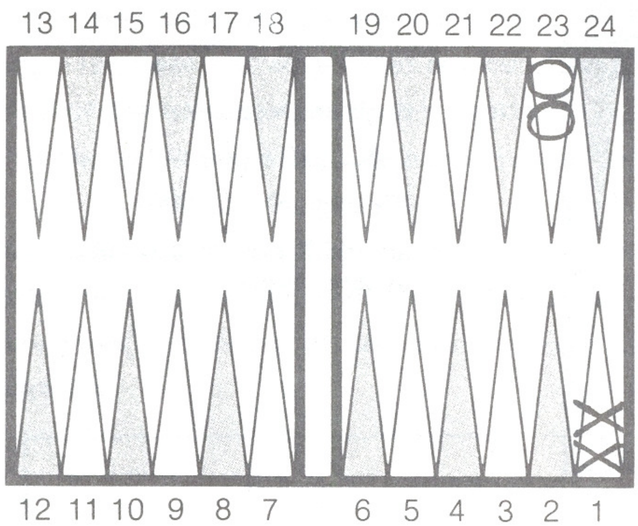
Example 27:



O gives the cube

Bet = $10 a point

X's capital = $1,000

Should **X** accept the cube at any level?

For **X**, we have P = 10/36 and Q = 26/36. For all the cube levels we have:

Level of the cube after being offered by O	Fraction bet if accepted F_a	Rate of increase $R_a = (1+A \times F_a)^P \times (1-F_a)^Q$ (A=1, P=10/36, Q=26/36)	Rate of increase if refused $R_r = 1-F_r$	X's cube decision
2	2%	0.99095	.99	Take
4	4%	0.98158	.98	Take
8	8%	0.96190	.96	Take
16	16%	0.91879	.92	Drop
32	32%	0.81757	.84	Drop
64	64%	0.54857	.68	Drop
128	128%	0.0	.36	Drop

X can afford to accept the cube up to level 8 and should refrain from taking it beyond this level.

* * * * *

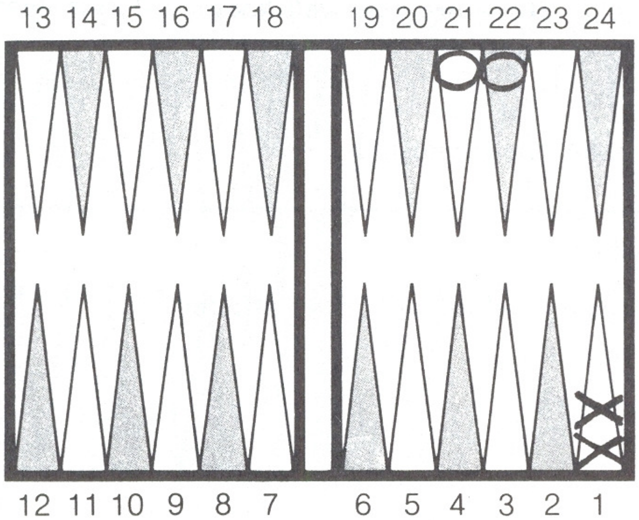
If you wish to maximize the rate of increase of your wealth, the Kelly theory, which is considered as the optimal one, advises that your criteria for accepting should be modified when the cube reaches high levels. Therefore, the well-known 3-1 rule does not automatically apply in all situations.

To decide if you should take or beaver a cube, the formula to use is:

If		$R_b > R_a \Rightarrow$ you should beaver
R_b	=	Average **Ratio** by which your capital will be multiplied after each trial if you **b**eaver
$>$	=	Greater than
R_a	=	Average **Ratio** by which your capital will be multiplied after each trial if you **a**ccept (see Formula 14)
$\Rightarrow$	=	It implies that
R_b	=	$(1 + A \times F_b)^P \times (1 - F_b)^Q$
A	=	Payoff $= \dfrac{\text{Amount you can win}}{\text{Amount you can lose}}$
F_b	=	Fraction of your capital wagered if you **b**eaver
P	=	Probability of winning
Q	=	Probability of losing (Q = 1-P)

Formula 15

Example 28:



O gives the cube

Bet = $10 a point

X's capital = $1,000

Should **X** drop, accept or beaver whatever the level of the cube?

For **X**, we have P = 19/36, Q = 17/36 and A = 1. **X** has to calculate R_r, R_a, R_b in relation to the level of the cube offered by **O.** Formulas 14 and 15 are used. The next table summarizes those calculations and the correct cube decision for **X.**

Level of the cube after being offered by O	Fraction bet by X if he accepts F_a	Rate of increase if refused R_r	Rate of increase if accepted R_a	Rate of increase if beavered R_b	X's cube decision
2	2%	.99	1.00091	1.00142	Beaver
4	4%	.98	1.00142	1.00124	Take
8	8%	.96	1.00124	.99601	Take
16	16%	.92	.99601	.96504	Take
32	32%	.84	.96504	.80143	Take
64	64%	.68	.80143	0.0	Take
128	128%	.36	0.0	0.0	Drop

So, if X's goal is to maximize the rate of increase of his capital, he should beaver from level 2 to 4, simply accept from level 4 up to level 64 and drop at level 128.

Note: The attentive reader will notice that this example is very similar to my incredible proposition. Indeed, in this example, **X**'s expectation is also 2/36 or 5.56% and the optimal bet for **X** is 5.56% of his capital.

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To help the reader visualize the effect of the Kelly theory on the taking strategy, a graph has been developed for the case of one-shot positions with no gammons involved. In this specific case, the standard strategy indicates that a player should take if his probability of winning is 25% or more and beaver if it is 50% or more, regardless of his bankroll or the level of the cube. The following graph which has been derived from Formulas 14 and 15 compares the standard strategy with the Kelly strategy.

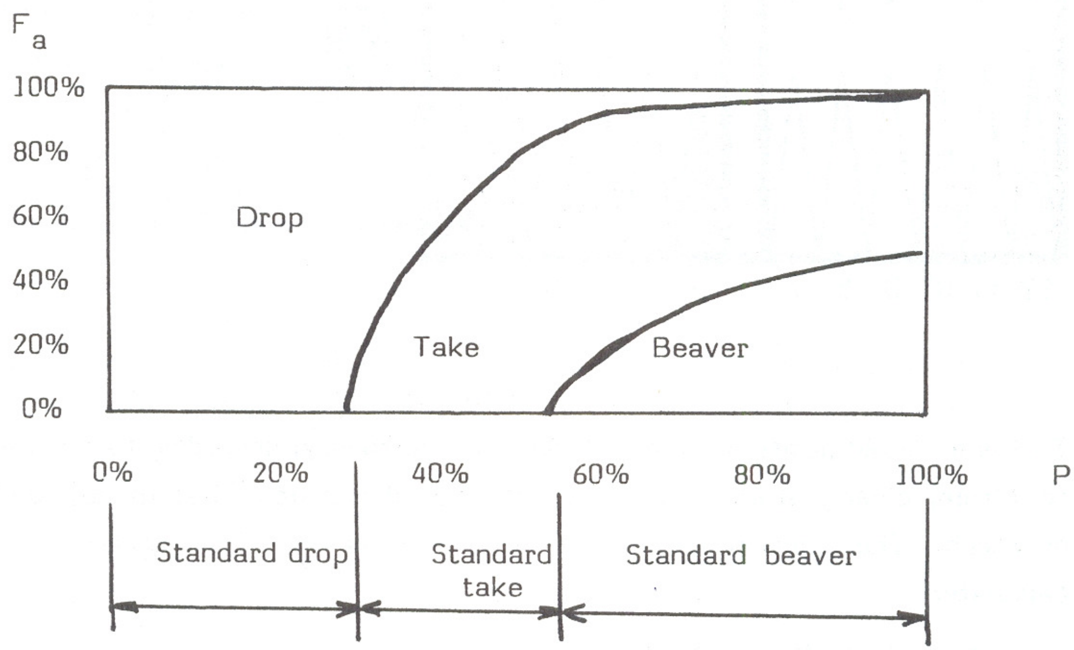


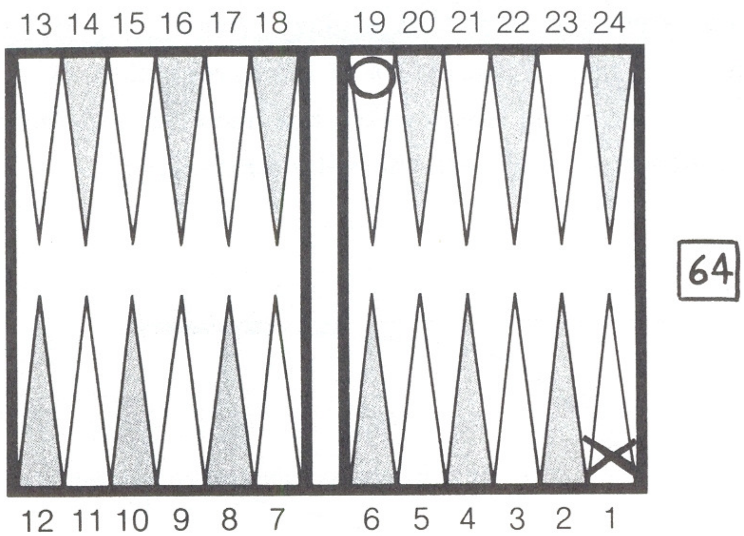
Diagram 5: Taking strategy for one-shot positions (with no gammons)

F_a = Your fraction bet if you accept
P = Your probability of winning

The reader interested in the mathematical derivation of Diagram 5 is referred to Appendix E.

It can be seen from the previous graph that the standard 3 to 1 taking strategy is a particular case of the Kelly strategy. When the fraction bet is low, the Kelly theory confirms the 3 to 1 theory. However, the higher the fraction bet, the more the Kelly strategy differs from the standard one. To illustrate, let us consider the following example.

Example 29:



O doubles
Bet = \$10 a point
X's bankroll = \$1,000
Should **X** take?

This is a classic position. The old conventional theory, according to the well-known 3 to 1 rule, clearly states that **X** has a marginal decision. Let us now use the Kelly approach. The fraction bet by **X** if he accepts is $F_a = 2\%$. Using Formula 14, we calculate:

$$R_r = 1 - F_a/2 = 0.99000$$

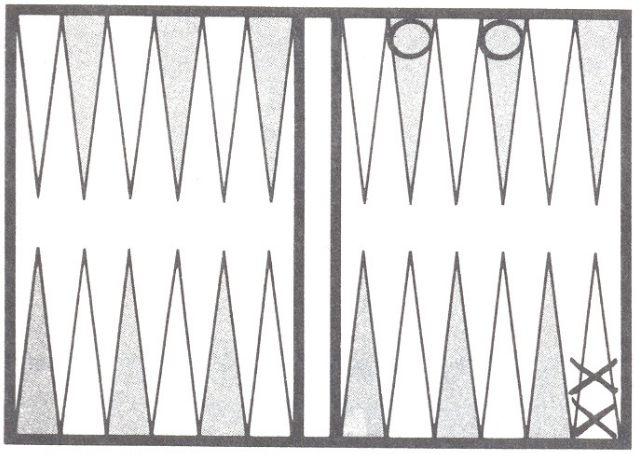
$$R_a = (1 + F_a)^{.25} \times (1 - F_a)^{.75}$$

$$R_a = (1.02)^{.25} \times (.98)^{.75} = .98985$$

Since the $R_r > R_a$, the cube should be dropped. It can be calculated that whatever the amount bet in this position, it is **never** a take for **X**.

* * * * *

Example 30:



O gives the cube

Bet = $1 a point

X's capital = $1,000

Should X drop, accept or beaver whatever the level of the cube?

O's winning probability is 14/36 (i.e. 6-6, 6-5, 6-4, 6-3, 5-5, 5-4, 5-3, 4-4 and 3-3) therefore, X's winning probability is 22/36 or 61.1%. Referring to Diagram 5, it can be seen that:

1) X should beaver if F_a is around 14% or less.
2) X should simply take if F_a is approximately between 14% and 90%.
3) X should drop if F_a is over about 90%.

Since the bet is $1 a point, we have:

Level of the cube after being offered by O	**Fraction bet by X if he accepts F_a**	**X's cube decision**
2	0.2%	Beaver
4	0.4%	Beaver
8	0.8%	Beaver
16	1.6%	Beaver
32	3.2%	Beaver
64	6.4%	Beaver
128	12.8%	Beaver
256	25.6%	Take
512	51.2%	Take
1024	102.4%	Drop

* * * * *

Note: The reader interested in further implications of the Kelly theory on the doubling cube may contact me at the address given in the conclusion.

CHAPTER 5: DILEMMA RELATED TO THE KELLY THEORY IN BACKGAMMON

Backgammon is probably the only game in which it is possible to increase the stakes during the game. The consequences are that the Kelly criterion can be applied in two distinct areas:

1) Establishing your optimal bet
2) Guiding your cube decisions.

This implies that:

1) The player using his optimal bet cannot have a "standard" cube behavior when the cube reaches high levels.
2) The player who plays with a low bet compared to his optimal bet can use rather conventional cube handling.

This brings about a dilemma which may be formulated as follows:

- Is it more profitable, in the long run, to play at the optimal bet level at the expense of a "special" cube handling, or to bet less and use the cube normally (using the standard approach)?

Example 31:

Consider the three following players:

1) Player A has an optimal bet of $50 a point. He usually bets $5 a point and handles the cube with the objective of maximizing his expectation. He is willing to risk all his capital on a single position as long as he is convinced he will maximize his expectation by doing so.

2) Player B always plays with his optimal bet of $50. He handles the cube in such a way as to maximize the rate of increase of his capital. In many instances, his cube behavior seems strange to observers unfamiliar with the Kelly criterion. About 80% of his cube decisions look normal whereas around 20% seem bizarre.

3) Player C plays with a bet representing 50% of his optimal bet, i.e. $25. He handles the cube in a way as to maximize the rate of increase of his capital. Over 90% of his cube decisions are standard.

Which one of these players will make more money in the long run?

Player A uses wrong principles of money management since 1) his bet is much too low and 2) he is willing to risk all his capital on a single game. Player B is trying to apply the Kelly criterion in the two areas (bet and cube). He has a better approach than Player A. Player C's approach is still superior to A's, but it remains to be asserted whether B's or C's approach is the best. However, they are both on the right track.

* * * * *

It seems your optimal bet should be established in such a way that 80% to 95% of all your cube decisions would correspond to the conventional theory which has as a goal to maximize the expectation of each game. Therefore, to solve this dilemma, we can multiply the optimal bet by a coefficient K_3. The coefficient K_3 as used in Formula 12 should probably vary between .5 and .9 as follows:

Table 3
K_3 coefficient

Approximate value of K_3	**Variables to consider**
.9	1) Your cube factor is very low (around 2.0) 2) Your opponent accepts credit 3) Your opponent likes settlements
.5	1) Your cube factor is high (higher than 3.0) 2) Your opponent insists on a cash payment 3) Your opponent never makes settlements

The analysis related to the best solution of the dilemma goes beyond the goal of this publication.

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CHAPTER 6: PRINCIPLES OF MONEY MANAGEMENT

6.1 The ten commandments of money management

The goal of this section is to enumerate the most important principles of money management that a backgammon player should always bear in mind. We will state the principles in order of importance and elaborate them afterwards in order to deduce a commandment. The commandments will then be summarized at the end of this section.

Principle no. 1: The established principles for games in general are necessarily applicable to backgammon.

So far, the theory of gambling has been formulated for money games in a general form. The most known principles are well presented in the book: The Theory of Gambling and Statistical Logic, by Richard A. Epstein (Academic Press) in Chapter 3 entitled: "Fundamental Principles of a Theory of Gambling". The principles of money management which are valid for money games are necessarily useful for backgammon even if some adjustments are required.

My first book entitled: Backgammon: How Much Should You Bet? explains several of those principles in Chapters 1 to 5 (incl.). This present book adds one principle. You should make an effort to understand and master these principles which are applicable to backgammon. Once you understand the principles, you should try to adapt them to practice. This adaptation should not necessarily be perfect, but it should be done. Any reasonnable adaptation of those principles will bring concrete results. Here is commandment no. 1:

YOU SHALL MASTER PRINCIPLES OF MONEY MANAGEMENT WHICH ARE APPLICABLE TO GAMES IN GENERAL AND ADAPT THEM TO THE SPECIFIC GAME OF BACKGAMMON.

* * * * *

Principle no. 2: A player using bad principles of money management will eventually lose money whatever his skill

To be the **favorite** in a game involving skill, you simply have to be more skillful than your opponent; but to be favored to **win money**, you not only have to be more skillful than your opponent, but also apply good principles of money management. Backgammon is a money game and if you do not manage your money correctly, you will eventually lose money. In Chapter 1 of this book, we have seen that a player betting out of proportion in a favorable game will eventually go broke. Even if such a statement may appear paradoxical, defying the theory will eventually turn against you, and this is valid whatever your skill level. The preceeding statement may be the source of the following question: "Is it more important to improve your principles of money management or to improve your skill?" To make money, you need both. But mastering principles of money management is a question of hours, and attaining a high skill level requires years. So here is the second commandment:

YOU SHALL BE AT LEAST AS CONCERNED WITH APPLYING GOOD PRINCIPLES OF MONEY MANAGEMENT AS IN TRYING TO IMPROVE YOUR SKILL.

* * * * *

Principle no. 3: Your long-term goal (and/or strategy) should be oriented towards making as much money as possible in the long run, even though your short-term objectives may differ

The backgammon player should always strive to make the most money in the long run. This goal is synonymous with playing to maximize the rate of increase of his capital. This may guide you on your day-to-day strategy.

In a certain way, the professional gambler may be compared to a businessman. The goal of a business is to make as much money as possible on a long-term basis but to reach this goal, the administrators may adopt different short-term strategies as:

- make as much money as possible on each sale
- make a reasonable amount of money on each sale
- make regular big sales to attract customers
- give the best possible service
- eliminate an "undesirable" competitor
- etc...

The backgammon player's short-term goal may be to play in order:

- to maximize the rate of increase of his capital
- to maximize his probability of success
- to obtain an established probability of success
- to maximize his hourly expectation
- to entertain his opponent
- etc...

Here is the third commandment:

YOU SHALL, IN THE LONG RUN, MAXIMIZE THE RATE OF INCREASE OF YOUR GAMBLING BANKROLL EVEN THOUGH YOUR SHORT-TERM GOAL MAY WELL BE DIFFERENT.

* * * * *

Principle no. 4: It may be good to pay for a lesson to improve your skill

The cheapest way to improve your skill is certainly to buy books, read them and re-read them. Nevertheless, it could be good from time to time to take a lesson from an expert. This lesson could be a "practical lesson", in other words, playing with him, preferably for low stakes, or a "theoretical lesson" (at approximately $50/hour).

While you are learning, i.e. you are playing to improve your skill, you should try to bet as low as possible. We could formulate commandment no. 4 as follows:

YOU SHALL BE WILLING TO PAY TO IMPROVE YOUR SKILL AS LONG AS YOU BELIEVE IT IS PROFITABLE IN THE LONG RUN.

* * * * *

Principle no. 5: Try to be involved in the most favorable game

Roughly speaking, the types of games could be divided as follows:

1) A game in which the expectation is constant and can be established mathematically. In such a game, you have:

   a) a favorable game if your expectation is positive
   b) a fair game if your expectation is nil
   c) an unfair game if your expectation is negative.

2) A game in which your expectation may not be calculated with a probabilistic approach (calculated mathematically), but only with a statistical approach (based on past results). The accuracy of your results mainly depends on the number of trials. The larger the number of trials, the more accurate the value of the expectation so obtained. Backgammon falls into that category.

Practically speaking, it is improbable that you will find a favorable game in the first category. Obviously, most of the games played in a casino offer the player a negative expectation. When the result is related to the player's skill, we automatically fall into the second category. The approach to use to calculate your player's expectation is explained in example 11 (see also Appendix A and B). To be able to correctly establish your player's expectation, you should 1) collect information on your opponents, and 2) be able to correctly approximate the expectation so obtained using the law of large numbers. Since there is presently no rating system in backgammon, you have to keep statistics on your opponents. Basing ourselves on the preceeding explanation, we can formulate the fifth commandment:

YOU SHALL HAVE ENOUGH DISCIPLINE TO RECORD ALL RESULTS, INTERPRET THEM, DERIVE YOUR "PLAYER'S EXPECTATION" AND SELECT THE OPPONENTS WHO GIVE YOU THE MOST FAVORABLE EDGE.

* * * * *

Principle no. 6: You should gamble with money you can afford to lose

To establish your gambling bankroll, you should consider your own means, not the means of your neighbour. Do not be influenced by the financial situation of your friends. We could formulate the sixth commandment as follows:

YOU SHALL ESTABLISH YOUR GAMBLING BANKROLL ACCORDING TO YOUR OWN FINANCIAL MEANS.

* * * * *

Principle no. 7: The unit bet is related to your gambling bankroll and to your goal

Your bet can be established according to your short-term goal, which could be:

1) to maximize the rate of increase of your capital (using Formulas 11 and 12 of this book)
2) to maximize your probability of success (Note *)
3) to obtain an established probability of success (Note *)
4) to maximize your hourly expectation (Note *).

Note *: See page 61 of the book: Backgammon - How Much Should You Bet?

Your appropriate bet will differ as you change your goal. Therefore, we can formulate commandment no. 7 as follows:

YOU SHALL DETERMINE YOUR APPROPRIATE BET ACCORDING TO YOUR GOAL(S) AND TO YOUR GAMBLING BANKROLL.

* * * * *

Principle no. 8: Your cube behavior is related to your goal

Your cube handling also depends on your goal. You should behave as follows:

1) If your goal is to maximize the rate of increase of your capital, you should handle the cube accordingly. In such a case, your cube handling depends on your bankroll and the old doubling cube theory is no longer valid. The players who are not convinced of this should seriously reconsider my "Incredible Chabot Paradoxical Proposition".

2) If your goal is to play to maximize your probability of success or to obtain an established probability of success, you should handle the cube in order to maximize your expectation.

3) If your goal is to maximize your hourly expectation, it should be reflected in your cube play. If your goal is to maximize your sessional expectation, you should behave accordingly with the cube.

Here is the eighth commandment:

> YOU SHALL HANDLE THE CUBE ACCORDING TO YOUR GOAL(S) AND TO YOUR GAMBLING BANKROLL.

* * * * *

Principle no. 9: You should never overbet i.e. bet out of proportion

In the context of this publication, the optimal bet is the one established using the Kelly system. For example, when the payoff is even, if your advantage is 5%, your optimal bet is 5% of your gambling bankroll. Betting out of proportion means betting well beyond the optimal bet established using the Kelly theory.

In backgammon, you can overbet in two cases:

1) When determining the unit bet at the beginning of the session, to be willing to bet 10% of your capital per game when your advantage is 3% is to overbet.

2) When giving the cube, to be willing to redouble and risk 20% of your bankroll when the advantage given by the position is only 5% is obviously to overbet (see my proposition).

The player who overbets is greedy. Control this emotion and never bet out of proportion. So here is the ninth commandment:

YOU SHALL NEVER BE GREEDY AND BET WELL BEYOND THE OPTIMAL BET ESTABLISHED USING THE KELLY CRITERION.

* * * * *

Principle no. 10: When you start losing your self-control and/or concentration, have the wisdom to end the session

Even if being involved in a favorable game goes in accordance with good principles of money management, it is also good advice to quit when you start losing your self-control or your concentration, since this would be equivalent to playing in an unfavorable game.

Since, on the one hand, you should not be willing to quit a favorable game, on the other hand, you should be willing to end the session if for any reason (loss of your self-control, loss of your concentration or loss of your session bankroll), you feel that the session becomes unfavorable. Here is the last commandment:

YOU SHALL HAVE THE WISDOM TO END THE SESSION IF FOR ANY REASON (LOSS OF SELF-CONTROL OR CONCENTRATION), YOU FEEL THAT THE SESSION BECOMES UNFAVORABLE.

* * * * *

THE TEN COMMANDMENTS OF MONEY MANAGEMENT IN BACKGAMMON

1) You shall master principles of money management which are applicable to games in general and adapt them to the specific game of backgammon.

2) You shall be at least as concerned with applying good principles of money management as in trying to improve your skill.

3) You shall, in the long run, maximize the rate of increase of your gambling bankroll even though your short-term goal may well be different.

4) You shall be willing to pay to improve your skill as long as you believe it is profitable in the long run.

5) You shall have enough discipline to record all results, interpret them, derive your "player's expectation" and select the opponents who give you the most favorable edge.

6) You shall establish your gambling bankroll according to your own financial means.

7) You shall determine your appropriate bet according to your goal(s) and to your gambling bankroll.

8) You shall handle the cube according to your goal(s) and to your gambling bankroll.

9) You shall never be greedy and bet well beyond the optimal bet established using the Kelly criterion.

10) You shall have the wisdom to end the session if for any reason (loss of self-control or concentration), you feel that the session becomes unfavorable.

* * * * *

Since it is a custom to have 10 commandments, I only dealt with the most important principles. Nevertheless, it would have been possible to extend the list with some sound but secondary principles, as for example the ones listed below.

1) Play to win, not for the thrill of the action.
2) If playing for high stakes, insist on cash payment after each game (or a small specified number of games).
3) Do not allow unlimited automatic doubles.
4) Try to reach settlements whenever a loss would hurt you.
5) Learn to "switch the gear" i.e. play as well at $5 or $10 or $20 a point.
6) Don't increase the bet when losing.

6.2 You are the stronger player

When you are the stronger player you may have different objectives. Three of them were dealt with in my book: Backgammon, How Much Should You Bet? on page 61. Another one is treated in this publication. Each goal has its merits and is not in contradiction with the others. Once you have chosen your goal, you have to establish your bet accordingly. Some possible goals are:

1) Playing to maximize the rate of increase of your capital. In this case, your optimal bet is established using Formulas 11 and 12 and your cube decisions are based on Formulas 13, 14 and 15.

2) Playing to maximize your probability of success; in this instance, your appropriate bet is the minimum one.

3) Playing to maximize your hourly expectation. You should try to bet as much as possible.

4) Playing to reach an established probability of success; in such a case your appropriate bet is obtained by using Formula 20 of the book: Backgammon, How Much Should You Bet?

If your goal is to maximize the rate of increase of your capital, it implies that:

A) Your bet should be established using Formulas 11 and 12. If the bet obtained is much greater than your opponent's usual bet, bet as high as he will accept (without going beyond your optimal bet).

B) You should encourage a credit basis (with selected players) because if your opponents accept checks or credit, your capital becomes your entire gambling bankroll. If, however, you agree to pay cash, your capital is limited to your session bankroll.

C) You should try to have a cube factor as small as possible which means:
. no automatic doubles;
. no "crazy cubes" (California rule);
. no beavers, no racoons, etc...

Note: Assuming your optimal bet is $20 (not adjusted by the cube factor), which of those alternatives would you prefer?

a) playing $20 a game with no cube;
b) playing $10 a point with a cube factor of 2.0 (points/game);
c) playing $ 5 a point with a cube factor of 4.0.

The best choice is A, followed by B and C. The reason is in accordance with the above mentioned statements.

D) You should encourage your opponent to accept settlements when the cube exceeds your optimal level (for instance, in example 21, you could pretend to give the cube at level 64 and insist for a correct settlement).

E) You should change your doubling cube handling in relation to the level of the cube and your opponent's point of ultimate take, which corresponds to the point where he has a marginal decision in taking. There are three possibilities:

1) The amount wagered once doubled and accepted is below or equal to your optimal bet (Diagram 2, page 11, when F is inferior to 5.56%). In this case, use the standard doubling theory.

2) This amount is between your optimal bet and the point where your rate of increase equals 1.0 (most of the time, this happens at twice your optimal bet) (in Diagram 2, when F is between 5.56% and 11.11%). Here, you should behave according to Formula 13.

3) This amount is superior to the point where R = 1.0 (in Diagram 2, when F is superior to 11.11%). In this case, wait until your opponent has gone beyond his point of ultimate take (i.e. he will drop) before doubling (unless you wish to play for the gammon).

6.3 You are the weaker player

In such a case, you may have three different goals:

1) Maximize your probability of success, which implies playing the minimum number of games with the largest possible bet. I refer you to the "Dany's example" of my letter of October 29th, 1982 which clearly demonstrates that "Dany" should play a 1-point match for $1,000 (this letter is enclosed with my book: Backgammon: How Much Should You Bet?).

2) Maximize your hourly expectation, i.e. minimize your hourly losses by betting as little as possible.

3) Maximize the probability that your opponent will go broke. In that case, the Kelly theory may be useful to give you some guidelines in establishing your bet.

If you select as an objective to maximize the probability that your opponent will go broke, then the incredible Chabot paradoxical proposition should inspire the weaker player on how to proceed. In fact, the weaker player should have a behavior opposite the stronger player's (which was developed in the preceeding section). If you are the weaker player, you should proceed as follows:

A) You should try to establish your bet in relation to your opponent's capital. The bet should go beyond twice your opponent's expectation multiplied by his capital. The more your opponent bets out of proportion, the more you are likely to make money. At the limit, you would like your opponent to bet all his capital on a single game.

B) You should insist on cash payment and reject checks, you should also have more money on you than your opponent.

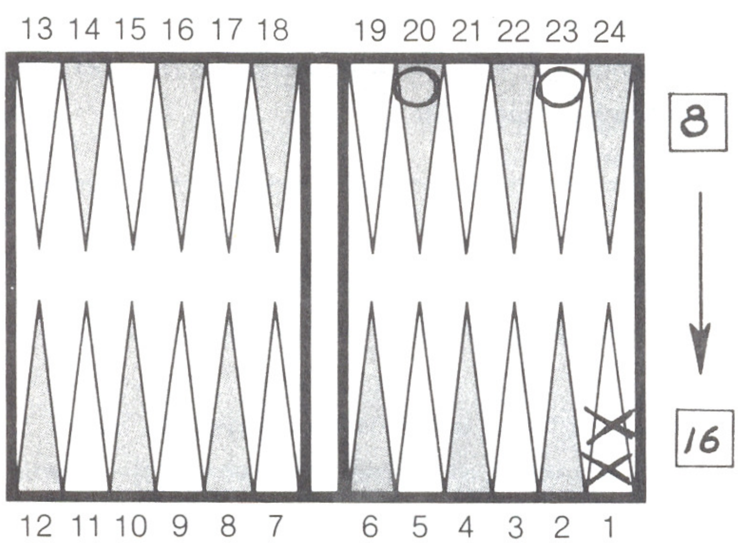
C) Your cube factor should be as large as possible. You should be willing to encourage unlimited automatic doubles, "crazy cubes", beavers, racoons, etc...

D) You should generally not be willing to make settlements.

E) You should be very wild with the cube, especially if the position is volatile. In some instances, it would be good to beaver in unfavorable positions (see: The Indredible Chabot Paradoxical Proposition).

When the weaker player's goal is to play to maximize the probability that his opponent will go broke, he can behave very strangely with the cube. But in such instances, the weaker player should know how much money his opponent has and make sure that he has more. You could imagine a millionaire playing at $100 a point against one of the Giant-32 (i.e. the best players in America) who would only have $3,000 on him. Such a millionaire could behave "strangely" with the cube if his goal is to maximize the probability that his opponent will lose his $3,000. Let us consider an example:

Example 32:



O gives the cube at level 16

Bet = $100 a point

O's capital = $3,000

X is a millionaire

X's goal is to maximize the chances that O goes broke

Should X drop, take or beaver?

We have seen through this publication that O's optimal bet is 2/36 x $3,000 = $166.67 i.e. $200. O should normally give the cube at level 2 and refrain from raising it beyond. By offering the cube at the 16-level, O is betting too high. If X's goal is to maximize the probability that his opponent goes broke, he should beaver.

* * * * *

CONCLUSION

Playing to maximize the rate of increase of your capital is one possible goal that you may wish to reach. Such a goal will permit you to 1) establish your optimal bet and 2) optimize your cube decisions.

The above mentioned objective is not in contradiction with the goals exposed in the book: Backgammon - How Much Should You Bet? which are summarized on page 61 as follows:

- playing to maximize your probability of success
- playing to maximize your hourly expectation
- playing to obtain a pre-established probability of success.

Once your goal has been established, you should determine your appropriate bet. As we have seen, the Kelly theory will maximize your chances of increasing your bankroll in the long run. This system will not only be useful in determining your optimal bet, but also in all the other areas of backgammon i.e. payment policies, rules concerning the cube, settlements and doubling cube handling. The following table summarizes the consequences of the Kelly system applied to backgammon.

Table 4

Consequences of the Kelly system on backgammon

	You are the stronger player	You are the weaker player
Your goal	Maximize the rate of increase of your capital	Maximize the probability that your opponent goes broke
Your bet	Established using Formulas 11 and 12	Twice beyond your opponent's optimal bet
Payment policy	Encourage credit and use checks	Insist on cash payment
Rules about the cube	Do not allow automatic doubles, "crazy cubes", beavers, etc.	Encourage automatic doubles, "crazy cubes", beavers, etc.
Settlements	Encourage your opponent to accept settlements	No settlements
Doubling cube handling	Function of your opponent's point of ultimate take when the cube is raised beyond your optimal level	Try to win the "big game" and be wild with the cube

Since playing to maximize your profits is the best long-term strategy, the Kelly system will surely soon be understood and used by most backgammon players.

It should be pointed out that the positions analyzed in this book were purposely kept simple. It is however possible to apply the Kelly criterion to any position, regardless of its complexity, which might be the subject of a future publication. Those of you who are interested in the more advanced implications of the Kelly system, or have any questions or problems to submit regarding this publication may contact me at:

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APPENDIX A: DANNY KLEINMAN'S ARTICLE CONCERNING EXPECTATIONS

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Using results to estimate expectations

"Suppose you wish to estimate your expectation (in points) per game or (equivalently) your stake-adjusted winning probability. The best estimates of probabilities are relative frequencies. The Law of Large Numbers says that as your experience becomes more and more extensive, your results tend to approximate your expectation RELATIVELY more closely. Though your cumulative net win or loss may tend to deviate more and more from its expected value, the RATIO of your cumulative net to the amount you wager approaches your expectation per unit wagered. To estimate your expectation, you must divide your cumulative net by your cumulative bet.

But how should you compute your cumulative bet? Michelin Chabot, in his book Backgammon: How Much Should You Bet?, tells us to count as the wager for each game, the level to which the doubling cube has risen by the end of the game. I disagree. By Chabot's logic, every double-in decreases your expectation as calculated by the ratio of results to wagers. To illustrate, let's suppose that you can either double your opponent in and have a 75% chance of winning the resulting 2-game; or wait, and have a 90% chance of winning a 1-game with the cube later. If you play 20 games and double, Chabot's methods yield a ratio of 20/40 = .50 by doubling but 16/20 = .80 by waiting. Yet clearly, it is better to win 20 points in 20 games than to win only 16 points.

Instead you should count the number of units at stake at the beginning of each game. Ordinarily, this is 1. But an "automatic" caused by each player rolling an equal die on the opening shake puts 2 units at stake. And in a chouette, you must multiply the initial stake (whether 1 or 2) by the number of opponents against whom you are wagering, if you are in the Box. If you do this, the ratio of your results to your cumulative bets will approach your per-game expectation.

Using this method, you will have some idea of whether---on the average---you figure to be a winner or a loser against a particular opponent or in a chouette with a particular set of opponents, and by how much.

But I caution you against using such "average" estimates. For relative frequencies are good estimates of underlying probabilities only when these probabilities are UNCHANGING. In fact, your probabilities and expectations in backgammon change almost constantly from one session to the next, and even within each session

Do you play less well as you get tired, or as your circadian rhythm lowers your brain temperature? This decreases your expectation as the session progresses---unless fatigue and the lateness of the hour affect your opponents even more.

Is your game improving? This increases your expectation from one session to the next---unless your opponents are improving more rapidly.

Let's assume that your own willingness to take chances is a constant, and that your cube actions are purely rational. But this is unlikely to be true of your opponents. They tend toward "steam doubles" and "steam takes" when they get behind on the scoresheet, and toward greater conservatism with the cube when they have a big "cushion".

Against players who ordinarily are neither wild nor overly tight with the cube, you will tend to do better when they are at either extreme on the backscore. You can "steal points" against the players who are ahead and win big games against the players who are behind. But against players ordinarily wild or tight with the cube, your results will vary in strange ways. You may have an edge against Colonel Whiteflag which disappears as soon as he starts to steam---for then he acts rationally, not overtimidly, with the cube. Conversely, your advantage against Diana Dialacube may disappear as soon as she piles up a big plus on the scoresheet and tightens up her cubes. In his heyday as a hustler, Hersch Malamud used to go so far as NOT to cut into her chouette if he saw that Diana was already a substantial winner.

In short, "average" expectations are meaningless to the extent that they mask individual differences stemming from significant variables."

APPENDIX B: ANSWER TO KLEINMAN'S ARTICLE

In Kleinman's book Double-sixes From The Bar, there is an article entitled: "Using Results to Estimate Expectations". You will find it reproduced in Appendix A.

Kleinman tried to demonstrate that my approach in establishing the player's expectation was incorrect. I clearly demonstrate in this appendix why my approach is valid and his theoretically incorrect. The concept of player's expectation is so important that you should understand why I have to clarify the confusion created by Kleinman. When we use our results to estimate our expectation, the expectation so obtained is obviously our "player's expectation" and not the "expectation of a specific backgammon position". I wonder if Kleinman makes a clear distinction between these two very different concepts. On the one hand, Kleinman agrees with my Formula 7 which is:

$$\text{Expectation} = \frac{\text{Cumulative Gains (or losses)}}{\text{Cumulative Wagered Amount}} \qquad \text{Formula 7}$$

but on the other hand, he disagrees with my way of establishing the cumulative wagered amount and contends that "You should count the number of units at stake at the **beginning** of each game". Since I disagree with him, I will begin by giving you some practical examples on how cumulative wagered amount should be calculated.

Example B1:

You play heads or tails 10 times in a row. You bet $5 on each game. What is your cumulative wagered amount?

Your cumulative wagered amount is $50 (10 games at $5 per game). It is necessary to pay attention to the fact that the cumulative wagered amount is not dependent on your results. Kleinman should certainly agree.

Example B2:

You play backgammon without cubes and no gammons (nor triple games). You bet $5 on each game and play 10. What is your cumulative wagered amount?

Your cumulative wagered amount is $50 (i.e. 10 games at $5 per game). Kleinman should also agree.

Example B3:

You play backgammon with the cube and with gammons possibilities. There are no automatic doubles. You played 10 games and obtained the results given in Example 11. What is your cumulative wagered amount?

As we saw in Example 11, the cumulative wagered amount is $100 (10 games at $10). Because Kleinman believes you should calculate your cumulative wagered amount taking into account only the level of the cube at the **beginning** of the game, he would have obtained the cumulative amount of only $50 (10 games at $5). We will see in the next example why Kleinman's approach is incorrect.

* * * * *

In Kleinman's book, Vision Laughs at Counting, there is an article entitled "The Kleinman Scale" (pages 155-157), in which he gave the following formula:

P	= (E + 1)/2	
P	= "probability of winning the game"	
E	= "expected value of the game to the player"	Formula B1

Since on page 157 of his Vision Laughs at Counting he gives an example where a player won 12 points in 40 games and concludes that "E", the expected value, is .3, I conclude that Kleinman sees "E" as follows:

$$E = \frac{\text{"expected value of the game to the player"}}{} = \frac{\text{number of points won (or lost)}}{\text{number of games played}} \qquad \text{Formula B2}$$

My Formula: P = .5 + E/2, corresponds exactly to the Formula B1; but I call "P" the single-trial probability of success instead of the "probability of winning the game". I also labelled the expectation "E" while Kleinman called it the "expected value of the game to the player". Now, in the article quoted in Appendix A, Kleinman would like to call "P" the "stake-adjusted winning probability". **Whichever way "P" is defined, it is a well-known fact that the probability of occurence of any event is always from 0% to 100%.**

I would like to point out that Formula B2 strongly differs from Formula 7. In Formula B2, the denominator is "number of games played" and in Formula 7, the denominator is the "cumulative wagered amount"; it follows that the results of Formula 7 are expressed in "percentage (%)" and the results of Formula B2 are expressed in "points per game". I really wonder if Kleinman made a clear distinction between the expectation concept (as calculated by Formula 7) and the notion of "expected value of the game" (as calculated by Formula B2).

Example B4:

You play 10 games (without automatics) with a very weak opponent and win 8 games at 4 points each and lose 2 games at 1 point each. You have, then, won 30 points in 10 games.

A) What would be Kleinman's value for "E" and "P"?

I don't know if, **now**, Kleinman would prefer to call "E" the "expected value of the game to the player" (ref. Vision Laughs at Counting, page 155) or the "expectation" (ref. Double-sixes From The Bar, page 117). According to Formula B2, he would have obtained an expected value of 3.0 (the 30 points won divided by 10 **games played**) and, according to his contention that the cumulative wagered amounts should be established by counting "the number of units at stake at the beginning of each game", he would also obtain an "expectation" of 3.0 (30 points won divided by 10 **points wagered**).

I don't know if, **now**, Kleinman would prefer to call "P" the "probability of winning the game" (ref. Vision Laughs at Counting, page 155) or the "stake-adjusted winning probability" (ref. Double-sixes From The Bar, page 116, 117); but according to Formula B1, he would obtain:

P = (E + 1)/2 = (3.0 + 1)/2 = 2.0 = **200%**

Since it is theoretically impossible to have a probability of more than 100%, Kleinman's approach is certainly incorrect.

B) What is your (player's) expectation and single-trial probability of success?

In my approach "E", your expectation is calculated by using Formula 7:

$$\text{Expectation} = \frac{\text{Cumulative Gains (or losses)}}{\text{Cumulative Wagered Amount}}$$

For this example, you won 30 points or $30 if the bet was $1 a point. Your cumulative wagered amount would be established as follows:

Number of games (won or lost) ending with 1 point (2 games x 1 point x $1 a point):	$ 2
Number of games (won or lost) ending with 4 points (8 games x 4 points x $1 a point):	$ 32
Amount wagered:	$ 34

Your expectation is established as follows:

$$\text{Expectation} = \frac{\text{Cumulative Gains (or losses)}}{\text{Cumulative Wagered Amount}} = \frac{\$30}{\$34} = 88\%$$

Using my approach, "P", your single-trial probability of success will be established as follows:

$$P = .5 + E/2 = 50\% + 88\%/2 = 50\% + 44\% = 94\%$$

C) What are your odds of winning?

Knowing "Pr", a probability of occurence of an event, the odds of occurence of this event is evaluated as follows:

$$\text{Odds of occurence} = \frac{Pr}{1 - Pr}$$

Pr = probability of occurence of an event

Formula B3

So, according to my approach, your odds of winning would be 94%/(100%-94%) = 94%/6% = 15.7 against 1. I really don't know what Kleinman would obtain since he would have calculated a probability of **200%** !!!...?

* * * * *

Since according to my approach your single-trial probability of success will always vary from 0% to 100%, I believe that my approach to establishing your expectation and single-trial probability of success is valid. I wonder if Kleinman would have written his article (reproduced in Appendix A) had he perfectly understood the difference between the concept of "player's expectation", "expectation of a specific backgammon position" and the "expected value of the game".

* * * * *

APPENDIX C: DERIVATION OF COEFFICIENT K_2

The notation we will use is:

P = Percentage of wins

Q = Percentage of losses, $Q = 1 - P$

E = Expectation, $E = P - Q$

R = Optimal rate of increase, $R = (1 + E)^P (1 - E)^Q$

N = Number of games played before the bet is multiplied by C

C = Ratio by which the stakes are multiplied after N games

Br_1 = Initial bankroll

Br_2 = Expected bankroll after N games, $Br_2 = R^N \times Br_1$

Ob_1 = Initial optimal bet, $Ob_1 = E \times Br_1$

Ob_2 = Expected optimal bet after N games, $Ob_2 = E \times Br_2$

B_1 = Actual bet at the beginning of the session, $B_1 = K_2 \times Ob_1$

B_2 = Actual bet after N games, $B_2 = C \times B_1$

K_2 = Coefficient 2, $K_2 = \frac{B_1}{Ob_1}$

We shall consider two practical cases: in the first, your opponent lets you decide of N and in the second, N is imposed. In both cases, it is assumed you are free to select B_1, the bet at the beginning of the session. We further assume that the value of C is fixed, and your opponent is not willing to change it. Another assumption is that your expectation E remains constant during the N games.

Case no. 1: You are free to select N

Since you are free to select N, you should choose it in such a way that after the stakes are increased, you will be betting your new optimal bet. We may write:

$$Ob_2 = C \times Ob_1$$

$$E \times Br_2 = C \times E \times Br_1$$

$$Br_2 = C \times Br_1$$

Let us call N* the optimal number of games to be played. We have:

$$R^{N*} \times Br_1 = C \times Br_1$$

$$\Rightarrow R^{N*} = C$$

$$\Rightarrow \ln(R^{N*}) = \ln C$$

$$\Rightarrow N* \ln R = \ln C$$

$N* = \frac{\ln C}{\ln R}$	Formula C1

Example C1:

You regularly play against an opponent with which your percentage of wins is 60%. He gives you the opportunity to decide of the amount of the bet and the number of games to be played before the bet is doubled. What should you select as the number of games?

We have: $E = P - Q = 0.6 - 0.4 = 0.2$

$$R = (1.2)^{0.6} \times (0.8)^{0.4} = 1.02034$$

$$C = 2$$

Using Formula C1:

$$N* = \frac{\ln C}{\ln R} = \frac{\ln(2)}{\ln(1.02034)} = 34.4$$

Therefore, your strategy is to play with your optimal bet and specify 35 as the minimum number of games to be played before the bet can be increased.

* * * * *

Case no. 2: The number of games N to be played before increasing the stakes is imposed

In this case, your objective is to wager your new optimal bet when N games are over. Thus, we can write:

$$Ob_2 = B_2$$

$$\Rightarrow E \times R^N \times Br_1 = C \times B_1$$

$$\Rightarrow B_1 = \frac{E \times R^N \times Br_1}{C} \qquad (1)$$

Since $K_2 = \frac{B_1}{Ob_1} = \frac{B_1}{E \times Br_1}$, we have

$$B_1 = K_2 \times E \times Br_1$$

Setting this value in formula (1), we obtain:

$$K_2 \times E \times Br_1 = \frac{E \times R^N \times Br_1}{C}$$

After simplifications, we have:

$K_2 = \frac{R^N}{C}, K_2 \leq 1.0$	Formula C2

Notes: 1) When N = 0, we have: $K_2 = \frac{R^0}{C} = \frac{1}{C}$

2) When N = N*, we derive:

$$K_2 = \frac{R^{N*}}{C} = \frac{R^{(\ln C/\ln R)}}{C}$$

$$\Rightarrow \ln K_2 = \ln\left[\frac{R^{(\ln C/\ln R)}}{C}\right]$$

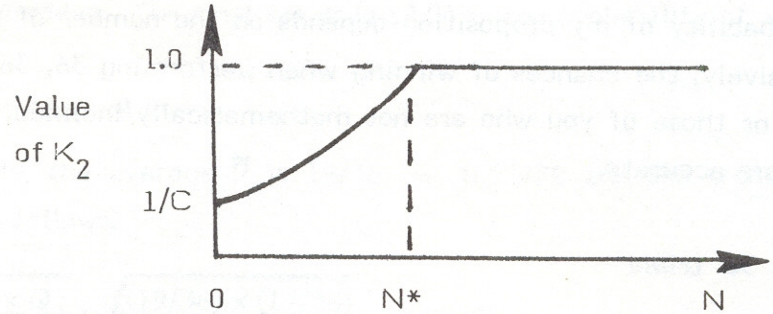
$$\Rightarrow \ln K_2 = \ln\left[R^{(\ln C/\ln R)}\right] - \ln C \qquad \text{(Since } \ln(\tfrac{a}{b}) = \ln a - \ln b\text{)}$$

$$\Rightarrow \ln K_2 = \left(\frac{\ln C}{\ln R}\right) \ln R - \ln C \qquad \text{(Since } \ln a^b = b \ln a\text{)}$$

$$\Rightarrow \ln K_2 = 0$$

$$\Rightarrow K_2 = 1.0$$

3) On a graph form:



It is noted that we arbitrarily affix an upper limit of 1.0 for the value of K_2 because this goes with the context of the Kelly criterion: to bet over the optimal bet is undesirable.

* * * * *

APPENDIX D: MATHEMATICAL ANALYSIS OF THE INCREDIBLE CHABOT PARADOXICAL PROPOSITION

The winning probability of my proposition depends on the number of trials. I will calculate, sucessively, the chances of winning when performing 36, 360, 1,000 and 10,000 trials. For those of you who are not mathematically inclined, I assure you that the results are accurate.

Case no. 1: N = 36 trials

The exact future opponent's capital and my exact future capital in relation to the number of wins are tabulated as follows:

Number of wins W	Exact opponent's future capital $(1.2)^W \times (.8)^{36-W} \times \$1,000$ (\$)	My exact future capital $(1.05)^W \times (.95)^{36-W} \times \$1,000$ (\$)	Results
19	719.41	1,056.57	I win
20	1,079.11	1,167.79	I win
21	1,618.66	1,290.71	I lose
22	2,427.99	1,426.58	I lose

It can be seen that when performing 36 trials, I will win if the results are inferior or equal to 20 successes, a success being defined as bearing off the two checkers on a given trial. If there are more than 20 successes, I will lose.

Therefore, to determine my probability of winning, I have to calculate the probability of obtaining from 0 to 20 successes in 36 trials. In order to do this, I will use two different approaches and compare the results.

1st approach: Binomial law

This law allows us to calculate the probability of occurence associated with each possibility. For example, the probability of obtaining exactly 20 successes in 36 trials when the probability of winning is 19/36, is calculated as follows:

$$\frac{36\,!}{20\,! \times 16\,!} \times (19/36)^{20} \times (17/36)^{16} = 12.56\%$$

Note: $36\,! = 36 \times 35 \times 34 \times \ldots \times 3 \times 2 \times 1$

The individual probabilities of obtaining from 0 to 20 successes are similarily computed and added. The final result is 69.04% (the probability of winning).

2nd approach: Normal law

With 36 trials, the average $\bar{P}$ is 19/36 = 0.52778 and the standard deviation is calculated as follows:

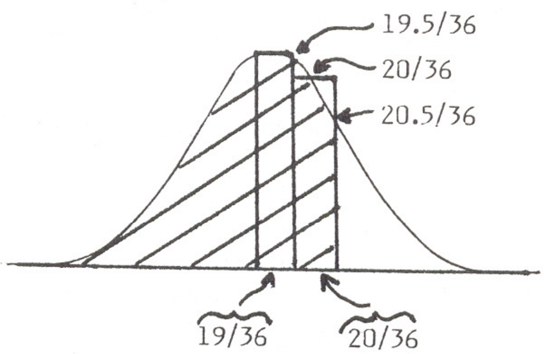
$$\sigma = \sqrt{\frac{P \times Q}{N}} = \sqrt{\frac{(19/36) \times (17/36)}{36}} = 0.08320$$

The probability that the proportion P is smaller than P_1 is equal to the probability that Z be smaller than $\frac{P_1 - \bar{P}}{\sigma}$.

It follows that:

$$Pr\,(P < P_1) = Pr\,(Z < \frac{P_1 - \bar{P}}{\sigma})$$

Because the normal curve is continuous and the actual distribution is discrete, it is necessary to make a small adjustment.



The probability of obtaining from 0 to 20 successes is calculated as follows:

$$Pr\,(P < 20.5/36) = Pr\,(Z < \frac{20.5/36 - 19/36}{.08320}) = 69.17\%$$

It can be seen that both approaches give very similar results (69.04% vs 69.17%). When "N" is below 30, it is usually preferable to use the binomial law, which gives more exact results. However, when P x N and Q x N are above 15, it is possible to use the normal law.

With 36 trials, the probability of winning is 69.1%. The expectation is calculated as follows:

$$E = P_1R_1 + P_2R_2 = 69.1\% \times \$1{,}000 + 30.9\% \times (-\$1{,}000)$$
$$= \$691 - \$309 = \$382$$

Since it will probably take two hours to explain the proposition, execute it and compile the results; you will see that my hourly expectation is around $200/hour.

Case no. 2: N = 360 trials

The future capital in relation to the number of wins is calculated as follows:

Number of wins W	Exact opponent's future capital $(1.2)^W \times (.8)^{360-W} \times \$1{,}000$ (\$)	My exact future capital $(1.05)^W \times (.95)^{360-W} \times \$1{,}000$ (\$)	Results
201	3,211.73	5,213.18	I win
202	4,817.59	5,761.93	I win
203	7,226.38	6,368.45	I lose
204	10,839.57	7,038.81	I lose

It can be concluded that in 360 trials, I will win if the results are inferior or equal to 202 successes, if not, I will lose $1,000. The calculations proceed as follows:

$$P = 190/360 = .52778$$

$$\sigma = \sqrt{\frac{P \times Q}{N}} = \sqrt{\frac{(19/36) \times (17/36)}{360}} = .02631$$

$$Pr\,(P < P_1) = Pr\,(Z < \frac{P_1 - P}{\sigma})$$

$$Pr\,(P < 202.5/360) = Pr\,(Z < \frac{202.5/360 - 190/360}{.02631}) = 90.7\%$$

With 360 trials, my probability of winning is 90.7%. My expectation is:

$$E = P_1R_1 + P_2R_2 = 90.7\% \times \$1{,}000 + 9.3\% \times (-\$1{,}000)$$
$$= \$907 - \$93 = \$814$$

Because it will take about four hours to explain the proposition, execute it and compile the results; you will see that my hourly expectation is approximatively $200/hour.

Case no. 3: N = 1,000 trials

The future capital in connection with the number of wins is tabulated as follows:

Number of wins W	Exact opponent's future capital $(1.2)^W \times (.8)^{1,000-W} \times \$1,000$ (\$)	My exact future capital $(1.05)^W \times (.95)^{1,000-W} \times \$1,000$ (\$)	Results
561	75,367	128,192	I win
562	113,051	141,685	I win
563	169,577	156,600	I lose
564	254,365	173,084	I lose

Therefore, with 1,000 trials, I will win if the results are inferior or equal to 562 successes. The probability of winning calculated by using the normal law is 98.6%.

My expectation is:

$$E = P_1R_1 + P_2R_2 = 98.6\% \times \$1,000 + 1.4\% \times (-\$1,000)$$
$$= \$986 - \$14 = \$972$$

Because it will take about 8 hours to explain the proposition, execute it and compile the results; you will see that my hourly expectation is approximately $125/hour.

Case no. 4: N = 10,000 trials

The future capital related to the number of wins is tabulated as follows:

Number of wins W	Exact opponent's future capital $(1.2)^W \times (.8)^{10,000-W} \times \$1,000$ (\$)	My exact future capital $(1.05)^W \times (.95)^{10,000-W} \times \$1,000$ (\$)	Results
5626	3.884×10^{24}	5.944×10^{24}	I win
5627	5.826×10^{24}	6.569×10^{24}	I win
5628	8.739×10^{24}	7.261×10^{24}	I lose
5629	1.311×10^{25}	8.025×10^{24}	I lose

If you are using a pocket calculator, the above mentioned calculations have to be performed with the help of logarithms.

Recall: 1) $A^x = e^{x \ln A}$ (note $e = 2.71828...$)

2) $A^x \times B^y = e^{(x \ln A + y \ln B)}$

Examples: 1) $4^6 = 4096$

$4^6 = e^{6 \ln 4} = 2.71828...^{(8.31776...)} = 4096$

2) $1.2^{562} \times .8^{438} \times \$1,000 = \$113,051$

$562 \ln 1.2 + 438 \ln .8 = 4.72784...$

$(e^{4.72784...}) \times \$1,000 = \$113,051$

3) $1.2^{5627} \times .8^{4373} \times \$1,000 = ?$

$5627 \ln 1.2 + 4373 \ln .8 = 50.11665...$

$(e^{50.1165...}) \times \$1,000 = \$5.826 \times 10^{24}$

The above mentioned calculations give astronomical amounts and it would be impractical to perform 10,000 trials due to the length of the compilation involved. I win if the results are inferior or equal to 5,627 wins (if $W \leq 5,627$). Using the normal law, we have $\bar{P} = 19/36$, $\sigma = 0.00499$ and the probability of obtaining from 0 to 5,627 successes with 10,000 trials is superior to 99.99999999% !

With 10,000 trials, my expectation is simply $1,000. Since the time for such a test is approximately 80 hours, my hourly expectation will be around $13/hour.

* * * * *

In summary, we have:

Number of trials performed	My probability of winning my proposition	My expectation	My hourly expectation
36	69.1%	$382	$200/hr
360	90.7%	$814	$200/hr
1,000	98.6%	$972	$125/hr
10,000	Almost 100%	$1,000	$13/hr

* * * * *

APPENDIX E: DERIVATION OF DIAGRAMS 4 AND 5

Diagram 4 is based on Formula 13, which states that if R_d R_{nd}, you should double. To derive the points on the curve, it is necessary to solve the equation $R_d = R_{nd}$, which can be written:

$$(1 + F_d)^P \times (1 - F_d)^Q = (1 + F_{nd})^P \times (1 - F_{nd})^Q$$

Since the diagram considers one-shot positions with no gammons, the payoff is A = 1 and does not appear in the equation. Noting that Q = 1-P and that $F_{nd} = 1/2 F_d$, the above equation becomes:

$$(1 + F_d)^P \times (1 - F_d)^{1-P} = (1 + 1/2 F_d)^P \times (1 - 1/2 F_d)^{1-P}$$

From this equation, F_d can be derived for a given P. However, the calculations are rather cumbersome and a trial-and-error method, performed by a computer or a programmable calculator, is the easiest approach. Using a such method, the following points for the diagram have been calculated:

P (%)	F_d (%)
50	0
52	5.3
55	13.3
57	18.6
60	26.6
65	39.7
70	52.6
75	65.1
80	76.9
85	87.5
90	95.8
95	99.8
100	100

The same method can be used to construct Diagram 5, derived from Formulas 14 and 15, where the equations to solve are:

$$(1 + F_a)^P \times (1 - F_a)^{1-P} = 1 - 1/2\, F_a$$

for the take-drop curve, and:

$$(1 + 2\, F_a)^P \times (1 - 2\, F_a)^{1-P} = (1 + F_a)^P \times (1 - F_a)^{1-P}$$

for the take-beaver curve. From these equations, the following points were calculated:

P (%)	F_a (%) for take-drop curve	F_a (%) for take-beaver curve
25	0	0
27	10.3	0
30	24.2	0
32	32.5	0
35	43.7	0
40	59.0	0
45	70.9	0
50	80.0	0
55	86.8	6.6
60	91.9	13.3
65	95.5	19.9
70	97.8	26.3
80	99.8	38.5
90	99.9	47.9
100	100	50

PHOTO GILLES ST-PIERRE

ABOUT THE AUTHOR

Michelin Chabot graduated from Sherbrooke University in the Province of Quebec (Canada) as a civil engineer in 1972. He played chess seriously from 1964 to 1976, but gave it up to devote himself entirely to backgammon.

This book will show you, among other things:

1) How to establish your optimal bet.
2) Why the standard 3 to 1 rule is now a thing of the past.
3) How to benefit from the ten-commandments of money management.
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