

ON DOUBLING IN TOURNAMENT BACKGAMMON*

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A strategy is determined which takes into account a player's skill, the score of the match, the value of the current game, and the possibility of winning a double-game (a gammon). The strategy described here has worked quite well in practice.

Introduction

In tournament backgammon, the objective is to win a specified number of points V before one's opponent wins V . (V is typically anywhere from 11 to 25.) Each game is initially worth 1 point. During the course of a 1-point game, either player may double the stakes before rolling his dice. If A doubles to 2, then B may either concede 1 point to A and start a new game or agree to play the existing game for 2 points. If B elects to play on then only he may redouble. Suppose that B decides to play and eventually redoubles to 4. Then A may either give B 2 points or play on for 4. If A decides to continue then only he may redouble, etc. There is no limit to the number of doubles. However, once the value of any game exceeds V doubling becomes irrelevant.

Gammons

It is possible for a player to win a gammon, or double-game, meaning that he wins $2K$ points if the value of the game is K . We will not consider the possibility of a triple-game or "backgammon" because these are rare.

The Crawford Rule

In tournament play the Crawford Rule is almost always used. This rule says that when either player reaches a score of $V - 1$ the other player may not double for one game. He may double after that game (assuming he wins it). Thus, suppose A needs 3 and B needs 4. A wins a 2 game, so he needs 1 and B needs 4. During the next game B may not double. If he wins, he will need 3 unless he gammons A and may double during any remaining game. Note that it is to B 's advantage to double immediately whenever possible when A needs only 1 point.

Continuous Time Markov Approximation

Let $p(t)$ denote the probability that player A will win a given game, assuming that neither player may double. If $p(t)$ is a continuous function of time, we call the game continuous. In backgammon, $p(t)$ is not continuous since it is defined only on nonnegative integers and may jump considerably from one move to the next. We begin by assuming that $p(t)$ is continuous and then make modifications to account for its actual discontinuity. The following lemma is from [1].

LEMMA 1. *In a continuous game, let the probability of A winning be x , and let $a, b > 0$ be such that $0 \leq x - a < x < x + b \leq 1$. Let E be the event that $p(t)$ becomes $x + b$ before it becomes $x - a$. Then the probability of E is $a/(a + b)$.*

Proof. We have, by conditioning on whether $p(t)$ first reaches $x + b$ or $x - a$, $x = p(E) \cdot (x + b) + (1 - p(E))(x - a)$. This yields

$$p(E) = \frac{x - (x - a)}{(x + b) - (x - a)} = \frac{a}{a + b}.$$

* Accepted by Melvin F. Shakun, former Departmental Editor for Planning, Game Theory and Gaming; received November 10, 1975.

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EXAMPLE 1. Suppose that $x = 0.75$. Then the probability that A wins ($p = 1$ before $p = 0$) is $0.75/1 = 0.75$.

EXAMPLE 2. Suppose that $x = 0.6$. Then the probability that $x = 0.8$ before $x = 0$ is $0.6/(0.6 + 0.2) = \frac{3}{4}$.

Lemma 1 may be extended to any linear function of $p(t)$, such as A 's expectation. In other words, the probability that A 's expectation rises by b before it falls by a is also $a/(a + b)$.

It is shown in [1] that in a continuous game A should double or redouble when $p = 0.8$. When A doubles, it does not matter whether B accepts or declines, his expectation in both cases is $-K$, where K is the value of the game before the double. To see this, observe that if B accepts, he will own the doubling cube at $2K$. He will win by reaching $p = 0.2$ before $p = 1$. Using Lemma 1, his probability of winning is $0.2/(0.6 + 0.2) = \frac{1}{4}$. Therefore, his expectation is $\frac{1}{4}(2K) - \frac{3}{4}(2K) = -K$. This is exactly what he would lose if he did not accept the double.

Note that A 's chances of winning when $p(t) = 0.8$ are 0.8 assuming that neither player may double. If B owns the cube then A 's chances are 0.75.

In a tournament game, the optimal doubling strategy for each player depends on the score of the match and the current value of the game. Let $A_i(x_1, x_2)$ denote player i 's chances of winning the match at the start of a new game when player 1 needs x_1 , player 2 needs x_2 , and $x_1 \geq 2, x_2 \geq 2$. When x_1 or $x_2 = 1$, it is possible that the player who is behind may not be able to double for one game because of the Crawford Rule. Let $B_i(x_1, x_2)$ denote player i 's chances when the underdog cannot double in the present game (but can in future ones). Let $\tilde{B}_i(x_1, x_2)$ denote player i 's chances when the underdog can double. Let y denote player 1's chances of winning a game and let gy denote his chances of winning a gammon. Then we have

$$\begin{aligned}
 B_1(1, 1) &= y, \\
 B_1(2, 1) &= gy + (y - gy)y, \\
 \tilde{B}_1(1, 1) &= y, \\
 \tilde{B}_1(2, 1) &= y \quad (\text{because player 1 will double immediately}), \\
 \tilde{B}_1(3, 1) &= gy + (y - gy)y \quad (\text{because player 1 will double immediately}), \\
 \tilde{B}_1(4, 1) &= gy + (y - gy)y \quad (\text{because player 1 will double immediately}),
 \end{aligned} \tag{1}$$

and, in general,

$$\begin{aligned}
 \tilde{B}_1(x_1, 1) &= gy\tilde{B}_1(x_1 - 4, 1) + (1 - g)y\tilde{B}_1(x_1 - 2, 1), \quad \text{and} \\
 B_1(x_1, 1) &= gy\tilde{B}_1(x_1 - 2, 1) + (1 - g)y\tilde{B}_1(x_1 - 1, 1).
 \end{aligned} \tag{2}$$

$B_1(1, x_1)$ and $\tilde{B}_1(1, x_1)$ are obtained from (1) and (2) by replacing y by $1 - y$.

When the state of the match is (x_1, x_2, K) (K represents the value of the current game), there will be some point in the current game at which player 1's (2's) advantage will be such that he can double and it will not matter whether his opponent takes or not. This point is called the *minimum take point* for player 2 (1). Let $D_1(x_1, x_2, K)$ denote player 1's chances of winning at the minimum take point for player 2, assuming that neither player may double. (In nontournament play, assuming the game is continuous, $D_i(-, -, K) = 0.80 \forall K$, and for $i = 1, 2$.) Let $\bar{D}_1(x_1, x_2, K)$ denote player 1's chances of winning at the minimum take point for player 2, given that his opponent may double. (In nontournament play $\bar{D}_1(-, -, K) = 0.75 \forall K$. *i.* whether the game is continuous or not.)

If a player's chances of winning after doubling are $\bar{D}(x_1, x_2, K)$ let $G_1 \cdot \bar{D}(x_1, x_2, K)$ denote his chances of winning a gammon. G_1 is a scalar which represents the fraction of wins that are gammons and will vary depending on the position. (In a straight race $G_1 = 0$.) Let $G_2 \cdot (1 - \bar{D}(x_1, x_2, K))$ denote the probability that the player accepting the double wins a gammon. In general, $G_1 > G_2$, since the player who accepts the double will often redouble instead of playing for the gammon.

When $x_1 > 2K$ and $x_2 > 2K$, $\bar{D}_1(x_1, x_2, K)$ is determined by the condition that player 1's probability of winning the match is the same whether player 2 takes or drops. Stating this mathematically, and letting \bar{D}_1 represent $\bar{D}_1(x_1, x_2, K)$, we have

$$G_1 \bar{D}_1 \cdot A_1(x_1 - 4K, x_2) + (1 - G_1) \bar{D}_1 \cdot A_1(x_1 - 2K, x_2) + G_2(1 - \bar{D}_1)A_1(x_1, x_2 - 4K) + (1 - G_2)(1 - \bar{D}_1)A_1(x_1, x_2 - 2K) = A_1(x_1 - K, x_2). \tag{3}$$

Solving (3) for \bar{D}_1 we obtain

$$\begin{aligned} & \bar{D}_1(G_1 A_1(x_1 - 4K, x_2) + (1 - G_1)A_1(x_1 - 2K, x_2) \\ & - G_2 A_1(x_1, x_2 - 4K) - (1 - G_2)A_1(x_1, x_2 - 2K)) \\ & = A_1(x_1 - K, x_2) - G_2 A_1(x_1, x_2 - 4K) - (1 - G_2)A_1(x_1, x_2 - 2K), \text{ or} \end{aligned} \tag{4}$$

$$\bar{D}_1(x_1, x_2, K) = \frac{A_1(x_1 - K, x_2) - G_2 A_1(x_1, x_2 - 4K) - (1 - G_2)A_1(x_1, x_2 - 2K)}{G_1 A_1(x_1 - 4K, x_2) + (1 - G_1)A_1(x_1 - 2K, x_2) - G_2 A_1(x_1, x_2 - 4K) - (1 - G_2)A_1(x_1, x_2 - 2K)} \tag{5}$$

A similar expression may be determined for \bar{D}_2 and is

$$\bar{D}_2(x_1, x_2, K) = \frac{A_2(x_1, x_2 - K) - G_2 A_2(x_1 - 4K, x_2) - (1 - G_2)A_2(x_1 - 2K, x_2)}{G_1 A_2(x_1, x_2 - 4K) + (1 - G_1)A_2(x_1, x_2 - 2K) - G_2 A_2(x_1 - 4K, x_2) - (1 - G_2)A_2(x_1 - 2K, x_2)} \tag{6}$$

When $x_1 < 2K$ and $x_2 > 2K$, player 2 will redouble immediately after player 1 doubles and rolls since player 1 needs at most $2K$ points. Therefore, when $x_1 < 2K$ and $x_2 > 2K$, (5) should be adjusted by substituting $2K$ for all occurrences of K , except in the term $A_1(x_1 - K, x_2)$, which remains unchanged. The same adjustment should be made to (6) except for the term $A_2(x_1, x_2 - K)$.

For x_1 or $x_2 \leq 2K$, we have $D_1(x_1, x_2, K) = \bar{D}_1(x_1, x_2, K)$, since no additional doubles will be made after the first one, except possibly an automatic redouble made by the player who is behind. (The equation $D_1 = \bar{D}_1$ is still correct for this case.)

Suppose now that $x_1 > 2K$, $x_2 > 2K$. Recall that $p(t)$ denotes player 1's chances of winning at time t assuming there is no cube. If player 1 doubles at $p(t) = D_1(x_1, x_2, K)$, he will have to reach $p(t) = 1$ to win before his opponent reaches $p(t) = 1 - (D_2(x_1, x_2, 2K) + \Delta)$. (The Δ is a correction factor to take into account the fact that the game is not continuous, and is a function of the board position.)

Therefore, using Lemma 1, and recalling that $\bar{D}_1(x_1, x_2, K)$ represents player 1's chances of winning after doubling at $p(t) = D_1(x_1, x_2, K)$, we have

$$\bar{D}_1(x_1, x_2, K) = \frac{D_1(x_1, x_2, K) - (1 - (D_2(x_1, x_2, 2K) + \Delta))}{D_2(x_1, x_2, 2K) + \Delta}, \text{ or}$$

$$D_1(x_1, x_2, K) = (D_2(x_1, x_2, 2K) + \Delta)(\bar{D}_1(x_1, x_2, K) - 1) + 1. \tag{7}$$

A similar expression may be obtained for $D_2(x_1, x_2, K)$.

If both players are equally skillful, the probability that player 1 reaches a doubling

position first when the match state is (x_1, x_2) is, by Lemma 1,

$$\frac{D_2(x_1, x_2, 1) + \bar{\Delta} - 0.5}{(D_1(x_1, x_2, 1) + \bar{\Delta} - 0.5) + (D_2(x_1, x_2, 1) + \bar{\Delta} - 0.5)}, \quad (8)$$

where $\bar{\Delta}$ is again a correction factor to account for discontinuities.

The probability that player 2 reaches a doubling position first is

$$\frac{D_1(x_1, x_2, 1) + \bar{\Delta} - 0.5}{(D_1(x_1, x_2, 1) + \bar{\Delta} - 0.5) + (D_2(x_1, x_2, 1) + \bar{\Delta} - 0.5)}. \quad (9)$$

When either player doubles, we will assume that his opponent drops. This assumption is accurate since the doubling points were determined by the condition that the probability of winning is the same whether one drops or takes.

Therefore, using (8) and (9), we have

$$= \frac{A_1(x_1, x_2)}{(D_2(x_1, x_2, 1) + \bar{\Delta} - 0.5)A_1(x_1 - 1, x_2) + (D_1(x_1, x_2, 1) + \bar{\Delta} - 0.5)A_1(x_1, x_2 - 1)} \quad (10)$$

A similar expression may be written for $A_2(x_1, x_2)$.

(1)–(10) and their counterparts for player 2 were combined recursively to determine $A_1(x_1, x_2)$, $D_1(x_1, x_2, K)$, $\bar{D}_1(x_1, x_2, K)$, $D_2(x_1, x_2, K)$, and $\bar{D}_2(x_1, x_2, K)$ for $K = 1, 2, 4, 8, 16$, and for $0 < x_1 < 25$, $0 < x_2 < 25$.

In order to do this, values for g , G_1 , G_2 , Δ and $\bar{\Delta}$ had to be determined. The following table was computed by taking $g = 0.25$, $G_1 = 0.25$, $G_2 = 0.15$, $\Delta = 0.08$, $\bar{\Delta} = 0.06$. These values were selected based on our playing experience plus some work on running games [4].¹

Table 1 may be read as follows. The x , respectively y , coordinate of Table 1 gives the number of points player 1, respectively player 2, needs to win. For each pair (x, y) , three numbers are given. The top number gives player 1's chances of winning the match ($A_1(x, y)$). The lower left number gives a modification of $D_1(x, y, 1)$ which takes gammon equity into account. Recall that when player 1 has a probability $D_1(x, y, 1)$ of winning, his expectation is considered to be $2G_1D_1 + (1 - G_1)D_1 - (1 - G_2)(1 - D_1) - 2G_2(1 - D_1)$, which when $G_1 = 0.25$ and $G_2 = 0.15$ equals $2.40D_1 - 1.15$. The number appearing in the lower left which we call $D_1'(x, y, 1)$ gives the probability player 1 would need in a *straight race* to have an expectation equal to $2.40D_1 - 1.15$. Since player 1's expectation in a straight race = $2D_1'(x, y, 1) - 1$, we have $D_1'(x, y, 1) = 1.2D_1(x, y, 1) - 0.075$. The number in the lower right is $D_2'(x, y, 1)$. \bar{D}_1' and \bar{D}_2' are similar modifications of \bar{D}_1 and \bar{D}_2 which take gammon equity into account.

Because of space limitations, we were not able to give all the results obtained by the program. For example, for the pair (4, 3) we have

$$\begin{array}{lll} D_1(4, 3, 1) = 0.670, & D_1(4, 3, 2) = 0.602, & D_1(4, 3, 4) = 1.0, \\ D_2(4, 3, 1) = 0.756, & D_2(4, 3, 2) = 0.820, & D_2(4, 3, 4) = 1.0, \\ \bar{D}_1(4, 3, 1) = 0.633, & \bar{D}_1(4, 3, 2) = 0.602, & \bar{D}_1(4, 3, 4) = 1.0, \\ \bar{D}_2(4, 3, 1) = 0.642, & \bar{D}_2(4, 3, 2) = 0.820, & \bar{D}_2(4, 3, 4) = 1.0, \\ D_1'(4, 3, 2) = 0.647, & D_2'(4, 3, 2) = 0.909, & \bar{D}_1'(4, 3, 1) = 0.685, \\ \bar{D}_2'(4, 3, 2) = 0.695, & \bar{D}_1'(4, 3, 2) = 0.647, & \bar{D}_2'(4, 3, 2) = 0.909. \end{array}$$

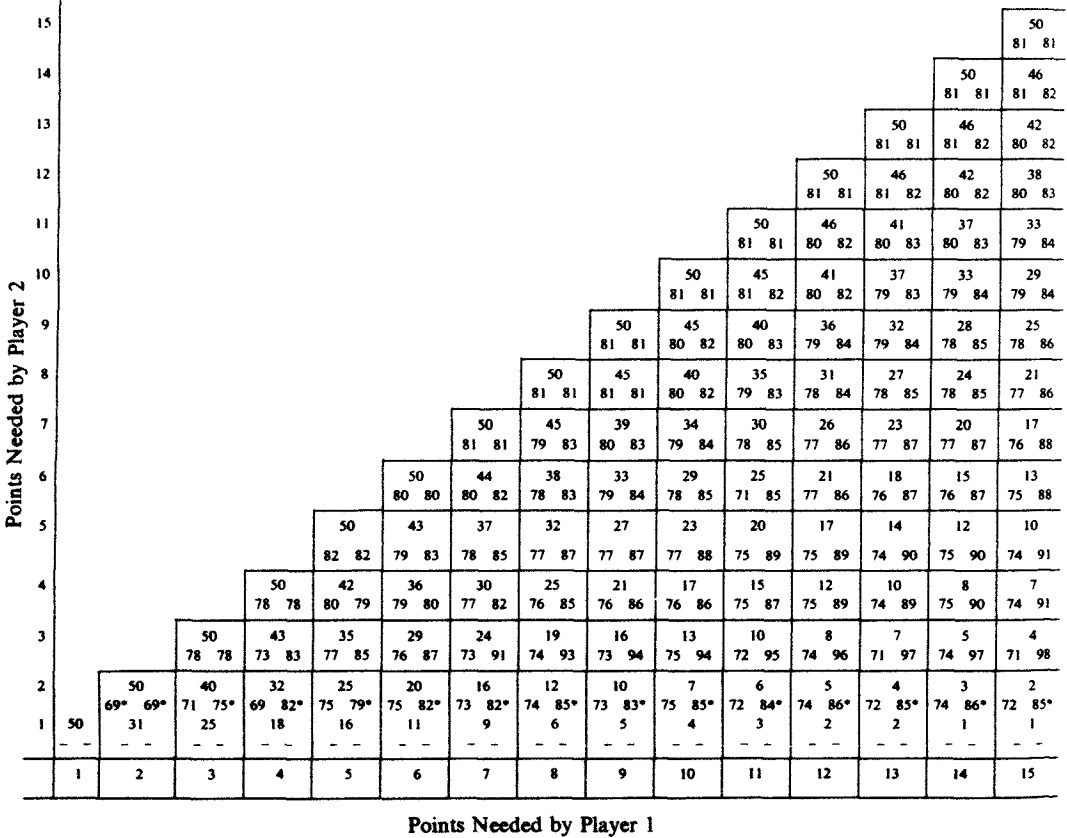
¹ In [4], we show that the effective doubling point in a straight race with 40 pips to go is approximately 0.90, which would mean that Δ in a straight race should be ≈ 0.10 . We take $\Delta = 0.08$ because gammons are being considered and the interval between the doubling points is smaller than would be the case in a straight race.

It is useful to observe that $\bar{D}_1(4, 3, 2) = A_1(2, 3)$, and $\bar{D}_2(4, 3, 2) = A_2(1, 4)$. Also, $D_1'(4, 3, 1) = 0.73$, which is considerably less than $D_2'(4, 3, 1) = 0.83$, even though $\bar{D}_1(4, 3, 1) \approx \bar{D}_2(4, 3, 1)$. This is because $D_1'(4, 3, 2) = 0.647$, while $D_2'(4, 3, 2) = 0.909$. In other words, player 1 can double earlier when the cube is at 1 because player 2 must wait a long time to double back when the cube is at 2.

TABLE 1

Results for a Match Between Two Players of Equal Skill

Top number in each box is player 1's chances of winning. Left (right) bottom number is minimum take point for player 2 (player 1). Take points have been normalized to include gammon equity (see text).



* Entry does not take gammon equity into account because of score. Figure given represents probability of winning the game.

Analysis When the Players Differ in Skill

The analysis for this situation is more complicated and we strive for a good approximate solution. Suppose that player 1 is the better player. Intuitively, the better player should travel from $p = 0.5$ to $p = 1$ with less effort, i.e., with effort proportional to $0.5/(1 + B)$, rather than 0.5, where $B > 0$. If we assume that the better player improves his position with less effort, while the inferior player requires the same effort as before, then a reasonable expression for the probability of the good player moving from x to $x + b$ before reaching $x - a$ would be

$$\frac{a}{a + b/(1 + B)} \tag{11}$$

Suppose instead that we assume that the better player moves more easily (with

effort proportional to $1/(1 + \alpha)$) whereas the inferior player moves with greater effort (proportional to $1/(1 - \alpha)$). A natural expression for the probability of player 1 reaching $x + b$ before $x - a$ would then be

$$\frac{a/(1 - \alpha)}{b/(1 + \alpha) + a/(1 - \alpha)} \tag{12}$$

This expression equals

$$\frac{a}{a + b(1 - \alpha)/(1 + \alpha)}$$

in other words it equals (11) when $(1 + \alpha)/(1 - \alpha) = 1 + B$. Thus, without loss of generality we will only consider formulas similar to (11).

To take the skill factor into account, (10) now becomes

$$A_1(x_1, x_2) = \frac{(D_2(x_1, x_2, 1) + \bar{\Delta}_2 - 0.5)A_1(x_1, -1, x_2) + (D_1(x_1, x_2, 1) + \bar{\Delta}_1 - 0.5)A_1(x_1, x_2 - 1)}{(D_2(x_1, x_2, 1) + \bar{\Delta}_2 - 0.5) + (D_1(x_1, x_2, 1) + \bar{\Delta}_1 - 0.5)/(1 + B)} \tag{13}$$

where $\bar{\Delta}_2 > \bar{\Delta}_1$, since the inferior player presumably makes less accurate doubles.

Effect of Skill on Minimum Take Points

A skillful player should be able, in general, to double earlier because his chances in most positions are greater than they would be against an equal player. Similarly, in a continuous game the weaker player should in general wait longer before doubling.

Determination of New Formulas for $D_1(x_1, x_2, K)$ and $D_2(x_1, x_2, K)$

We will assume that player 1 is the better player, and that his advantage is measured by $1 + B$. The x_1, x_2 in $D_1(x_1, x_2, K)$ and $D_2(x_1, x_2, K)$ will be suppressed. In other words, we will let $D_1(K)$ denote the minimum take point for player 2 when player 1 doubles from K to $2K$.

The formulas for $\bar{D}_1(K)$ and $\bar{D}_2(K)$ are the same as before and are given by (5) and (6). To obtain the new formula for $D_1(K)$, we note that when $p = D_1(K)$, player 1 must travel a distance $1 - D_1(K)$ to win before his opponent travels a distance $D_1(K) - (1 - D_2(2K) - \Delta_2)$. Thus we have

$$\begin{aligned} \bar{D}_1(K) &= \frac{D_1(K) - (1 - D_2(2K) - \Delta_2)}{D_1(K) - (1 - D_2(2K) - \Delta_2) + (1 - D_1(K))/(1 + B)} \\ &= \frac{(D_1(K) + D_2(2K) + \Delta_2 - 1)(1 + B)}{(D_1(K) + D_2(2K) + \Delta_2 - 1)(1 + B) + 1 - D_1(K)} \\ &= 1 - \frac{1 - D_1(K)}{(D_1(K) + D_2(2K) + \Delta_2 - 1)(1 + B) + 1 - D_1(K)} \tag{14} \end{aligned}$$

$$(\bar{D}_1(K) - 1)((D_1(K) + D_2(2K) + \Delta_2 - 1)(1 + B) + 1 - D_1(K)) = D_1(K) - 1, \text{ or}$$

$$\begin{aligned} 1 + (\bar{D}_1(K) - 1)((D_2(2K) + \Delta_2 - 1)(1 + B) + 1) \\ = D_1(K)((1 - \bar{D}_1(K))(1 + B) + \bar{D}_1(K) - 1 + 1), \end{aligned}$$

hence

$$1 + (\bar{D}_1(K) - 1)(D_2(2K) + \Delta_2 + B(D_2(2K) + \Delta_2 - 1)) = D_1(K)(1 + B(1 - \bar{D}_1(K))),$$

so

$$D_1(K) = \frac{1 + (\bar{D}_1(K) - 1)(D_2(2K) + \Delta_2 + B(D_2(2K) + \Delta_2 - 1))}{1 + B(1 - \bar{D}_1(K))} \tag{15}$$

TABLE 2

Results for a Match in Which Player 1 Has an Expectation of 0.4 of a Point Per Game (in Nontournament Play)

Top number in each box is player 1's chances of winning. Left (right) bottom number is minimum take point for player 2 (player 1). Take points have been normalized to include gammon equity (see text).

Points Needed by Player 2	15	100 --	99 88*70	99 99 70	98 94 72	97 93 72	96 90 73	94 89 74	93 88 75	91 87 76	89 87 77	86 86 77	84 85 78	81 85 78	79 84 79	76 84 79
	14	100 --	99 89*73	99 98 72	98 93 73	96 92 73	95 90 74	93 89 74	91 88 75	89 87 76	86 86 77	84 85 77	81 85 78	78 84 79	75 84 79	72 83 80
	13	100 --	99 88*70	98 98 70	97 92 72	95 92 73	93 89 74	91 88 75	89 87 76	86 86 77	83 85 78	80 85 78	77 84 79	74 84 79	71 83 80	68 82 81
	12	99 --	99 89*73	97 97 72	96 92 73	94 91 73	91 89 74	89 88 75	86 87 76	83 86 77	80 85 78	77 84 78	74 84 79	70 83 80	67 82 80	64 82 81
	11	99 --	98 87*71	96 96 70	94 90 73	92 90 74	89 88 75	86 87 76	83 86 78	80 85 78	76 84 79	73 83 80	69 83 80	66 82 81	62 82 81	59 81 82
	10	99 --	97 88*73	95 95 73	92 90 73	89 89 74	86 87 76	83 87 77	79 85 78	76 84 78	72 83 79	68 83 79	64 82 80	61 82 81	57 81 82	53 81 82
	9	98 --	96 87*72	93 95 72	90 89 74	86 89 76	83 87 77	79 85 78	75 84 79	71 83 80	67 83 81	63 82 81	59 81 82	55 81 83	51 80 83	48 80 84
	8	97 --	94 89*72	91 95 72	87 89 73	83 88 75	78 87 76	74 85 77	70 83 80	66 83 79	61 82 80	57 82 81	53 81 81	49 80 82	45 80 83	42 79 84
	7	96 --	92 85*72	88 92 72	83 86 75	79 87 77	74 84 78	69 83 80	64 82 81	60 81 82	55 81 82	51 80 83	47 80 84	43 79 85	39 79 86	36 78 86
	6	95 --	89 85*73	84 88 73	79 83 75	73 85 77	68 83 78	63 82 79	58 81 81	53 80 81	48 80 82	44 79 83	40 79 84	36 78 85	33 78 85	29 77 86
	5	91 --	85 82*75	78 86 77	73 82 79	67 83 81	61 82 82	55 79 84	50 79 86	46 78 87	41 79 87	37 77 88	33 77 88	30 76 89	26 77 89	23 75 90
	4	89 --	79 86*65	71 85 69	65 82 74	58 82 76	52 82 77	46 79 79	41 79 81	37 79 82	32 78 83	29 77 84	25 77 86	22 76 86	19 76 88	17 76 88
	3	84 --	72 79*69	64 79 76	57 76 82	50 78 85	44 78 86	38 75 89	33 76 91	29 75 93	25 76 93	22 73 94	19 76 95	16 73 96	14 75 96	12 73 98
	2	78 --	62 72*66*	53 73 71*	45 73 78*	38 76 76*	31 78 78*	27 75 78*	22 77 82*	19 75 80*	16 77 82*	13 74 81*	11 76 83*	9 74 82*	8 76 84*	6 74 83*
	1	60 --	42 --	36 --	28 --	25 --	19 --	17 --	13 --	11 --	9 --	8 --	6 --	5 --	4 --	3 --
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Points Needed by Player 1

* Entry does not take gammon equity into account because of score. Figure given represents probability of winning the game.

Performing a similar analysis and noting that

$$1 - \bar{D}_2(K) = \frac{1 - D_2(K)}{1 - D_2(K) + (D_2(K) - (1 - D_1(2K) - \Delta_1))/(1 + B)}, \text{ we obtain}$$

$$D_2(K) = 1 + \frac{(D_1(2K) + \Delta_1)(\bar{D}_2(K) - 1)}{B\bar{D}_2(K) + 1} \tag{16}$$

To incorporate the above formulas into a computer program, we assumed that the more skillful player would be a 6 to 4 favorite in a game with no cube, and that his expectation would approximately double in a money game with a cube (his expectation would be approximately 0.4 of a point per game). These assumptions allowed us to compute *B* in the following manner.

In a money game, $\bar{D}_1(K) = \bar{D}_2(K) = 0.75$. Substituting $\frac{3}{4}$ for $\bar{D}_1(K)$ and $\bar{D}_2(K)$ in (15) and (16), and taking $\bar{\Delta}_1 = 0.05$, $\bar{\Delta}_2 = 0.07$, $\Delta_1 = 0.07$, $\Delta_2 = 0.09$, we were then able to combine formulas (13), (15) and (16) to determine that value of B which made player 1's expectation in a money game equal to 0.4. Upon running the program with different parameter values, it became clear that the guesses of Δ_1 , Δ_2 , $\bar{\Delta}_1$, and $\bar{\Delta}_2$ were relatively unimportant compared to the assumption that player 1's expectation would be 0.4. In other words, the numbers in Table 2 are not affected significantly by small changes in Δ_1 , Δ_2 , $\bar{\Delta}_1$, $\bar{\Delta}_2$, but are affected significantly by a change in player 1's expectation.

Table 2 may be read in the same fashion as Table 1. For example, the three numbers in the box corresponding to (4, 3) indicate that when player 1 needs 4 and player 2 needs 3, player 1's chances are 57%, that player 2's minimum take point in a straight race is 76%, and that player 1's minimum take point is 82%. This does not contradict the fact that the underdog should generally double with weaker positions. The assumption is that player 2 plays poorly, so that when player 1 takes with a position that would give him an 18% chance against a good player (with no cube), his actual chances are 39.7%,² whereas when player 2 takes with a position that would give him a 24% chance against an equal player (with no cube), his actual chances are 20.3%.^{3, 4}

² This number is $1 - \bar{D}_2(4, 3, 1)$.

³ This number is $1 - \bar{D}_1(4, 3, 1)$.

⁴ The author would like to thank Gary Kobliska for helpful comments.

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