Take-Points in Money Games

by Rick Janowski

Guidance on doubling strategy in backgammon is provided by the following two theoretical models:

1. ***Dead-Cube Model*** **–** the *classical* model which makes no allowance for cube ownership.

2. ***Live-Cube Model*** – the *continuous* model which assumes maximum possible cube ownership value.

The former generally overestimates take-points and underestimates doubling-points (25% and 50% respectively, assuming no gammons). Conversely, the latter model underestimates take-points and overestimates doubling-points (20% and 80% respectively assuming no gammons). When considered together, however, they provide an envelope in which correct cube action decisions are to be found.

Dead-Cube Model

The owner of the cube is not afforded any additional benefits by it – he can neither double out his opponent nor raise the stakes at an opportune time. Effectively, the game is played out to its conclusion cubeless (but at the stake raised by the previous double). Consequently, take-points can be readily established from the risk-reward ratio.

Assume a double occurs in a game where, if played to conclusion, both players will win a mixture of single-games, gammons and backgammons. The effects of gammons and backgammons can be dealt with by introducing the following two variables for the player making the cube action decision (in this case, the player doubled):

W = Average cubeless value of games ultimately won

L = Average cubeless value of games ultimately lost

Consequently, a take would risk $2L-1$ points to gain $2W+1$ points. The minimum cubeless probability for a correct take (TP) is therefore:

$$TP = \frac{(2L-1)}{(2W+2L)} = \frac{(L-0\bullet 5)}{(W+L)} \qquad \text{…equation (1)}$$

This formula is also applicable when the data considered represents *effective* game winning chances.

Live-Cube Model

The owner of the cube is guaranteed to use the cube with optimal efficiency if he redoubles, at which point his opponent will have an optional pass/take. All subsequent redoubles by either of the two players are similarly optimal. There are, in fact, an infinite number of different possible live cube models identifiable by the following two variable factors:

1. The number of possible subsequent optimal redoubles. This can vary between unity and infinity. The infinite model is a good approximation to any of the finite models – all odd-numbered finite models give slightly higher cube-ownership values, whilst the even-numbered models give slightly lower ones. The discrepancy reduces progressively towards infinity. The relationship can be imagined as a dampened-sinusoidal curve with the infinite model as its axis. The man on the six-point versus man on the six-point position is an example of the s*ingle-subsequent redouble live model* (take-point = 18.75%). In fact, this live-cube model is the only one that exists in practice.

2. The change in gammon (and backgammon) rates throughout the life of the game. In most *real* backgammon positions, a player's rate of winning gammons will decrease when his opponent redoubles. A typical example is when a shot is hit in an ace-point game, which subsequently gives the opponent little, if any, gammon risk. The same general reduction in gammon rate will normally occur in the live cube models, as the greater the number of subsequent optimal redoubles, the higher the chance that one or both players will, at some point, take men off. The rate of gammon loss could be linear (e.g., % loss per opponent's redouble), or otherwise.

Assuming an infinite possible number of subsequent optimal redoubles, and a constant gammon rate (*W* and *L* are constant) for the sake of simplicity, the following formula was obtained, after some detailed mathematical analysis:

$$TP = \frac{(L - 0 \bullet 5)}{(W + L + 0 \bullet 5)} \qquad \text{…equation (2)}$$

Amazingly, the equation has a *simple* form. But what about the reduction in gammon rate, so far ignored? I investigated several different *reducing* models hoping to find that the above formula would still provide a reasonable estimate. What I found was much better; the formula is correct regardless of the gammon reduction rate considered, provided the *W* and *L* values used are *average* as opposed to *initial* ones! I wondered about this surprising result for some time and developed the following argument to support it:

What is the difference, in terms of risk and reward, between the live and dead-cube models? There are additional benefits from holding the cube which ***add*** to the basic dead-cube reward ($2W + 1$). What are they and when do they occur? They occur on the point of redoubling when the redoubler's equity jumps from 1·0 ppg (dead-cube) to 2·0 ppg (owning a 2-cube), a bonus of 1.0 ppg. (This is not strictly true, as the *dead-cube* equity is a little higher than 1·0 ppg, but this effect is balanced out by the equity jump occurring in more games than the cubeless take-point.) Consequently, if we add this *bonus* to the reward used in equation (1) for the dead-cube model, we arrive at equation (2) for the live cube model. As this argument is independent of any considerations of reducing gammon-rates, they would, indeed, appear to be irrelevant.

General Cube Model

Equations (1) and (2) above represent the take-point envelope in which correct take-points are to be found (the one known exception being the *man on the six-point versus man on the six-point* position). In any given position, the true take-point could be assessed by interpolating between the dead and live values, based on some intermediate value of cube-life, calculated, estimated, or just plain guessed at. The general form of these equations, given below again for clarity, allows a more elegant solution:

$$TP_{dead} = \frac{(L - 0 \bullet 5)}{(W + L)} \qquad \text{…equation (1)}$$

$$TP_{live} = \frac{(L - 0 \bullet 5)}{(W + L + 0 \bullet 5)} \qquad \text{…equation (2)}$$

Notice that the only difference is in the equations' denominators, with the *live* value having the additional *bonus* from cube-ownership, explained before. As this bonus represents the expected equity jump, it is proportional to the degree of cube-life of the position (and inversely proportional to its long-term volatility). Intermediate models can, therefore, be represented by a cube-life index, ***x***, which varies between 0·0 (dead cube, maximum volatility) and 1·0 (live-cube, zero volatility). The general form of equations (1) and (2) above is, therefore:

$$TP_{general} = \frac{(L - 0 \bullet 5)}{(W + L + 0 \bullet 5x)} \qquad \text{…equation (3)}$$

Clearly the value of ***x*** varies from position to position, and will commonly be different for both sides. Some of the important factors that determine its value include:

1. The distance from the target – the further away from the optimal doubling point you are, the less likely you are to hit the *bull's-eye*.

2. The size of the target – the size of the doubling window governs the size of the *bull's-eye*.

3. The relative movement between the shooter and the target – the volatility of the position governs the likelihood of hitting the *bull's-eye*, or even finding it, for that matter.

Finding accurate values for ***x*** is a difficult, almost impossible, task. However, we can make estimates of *typical* values for *typical* situations. In my opinion, for the majority of *typical* positions, ***x*** will commonly be between about ½ and ¾, with ⅔ being a *normal* value.

Cube Action Tables

To provide guidance on cube action, and to enable the reader to inspect the general results, the following tables are included:

Tables 1a, 1b, 1c – Cubeless take-points (for varying values of ***W*** and ***L***) for ***x*** values of 0·0 (*dead*), 1·0 (*live*), and ⅔ (*normal*).

Tables 2a, 2b, 2c – Cubeless take-equities (for varying values of ***W*** and ***L***) for ***x*** values of 0·0 (*dead*), 1·0 (*live*), and ⅔ (*normal*).

Cubeless take-equities (E_{take}) are calculated from the following general formula:

$$E_{take} = TP(W + L) - L \qquad \text{…equation (4)}$$

Cubeless Take-Point Tables

Table 1a Dead $(x = 0 \cdot 0)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	25·0%	22·2%	20·0%	18·2%	16·7%
cubeless	***1·25***	33·3%	30·0%	27·3%	25·0%	23·1%
loss	***1·50***	40·0%	36·4%	33·3%	30·8%	28·6%
value	***1·75***	45·5%	41·7%	38·5%	35·7%	33·3%
L	***2·00***	50·0%	46·2%	42·9%	40·0%	37·5%

Table 1b Live $(x = 1 \cdot 0)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	20·0%	18·2%	16·7%	15·4%	14·3%
cubeless	***1·25***	27·3%	25·0%	23·1%	21·4%	20·0%
loss	***1·50***	33·3%	30·8%	28·6%	26·7%	25·0%
value	***1·75***	38·5%	35·7%	33·3%	31·3%	29·4%
L	***2·00***	42·9%	40·0%	37·5%	35·3%	33·3%

Table 1c Normal $(x = \frac{2}{3})$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	21·4%	19·4%	17·6%	16·2%	15·0%
cubeless	***1·25***	29·0%	26·5%	24·3%	22·5%	20·9%
loss	***1·50***	35·3%	32·4%	30·0%	27·9%	26·1%
value	***1·75***	40·5%	37·5%	34·9%	32·6%	30·6%
L	***2·00***	45·0%	41·9%	39·1%	36·7%	34·6%

Cubeless Take-Equity Tables

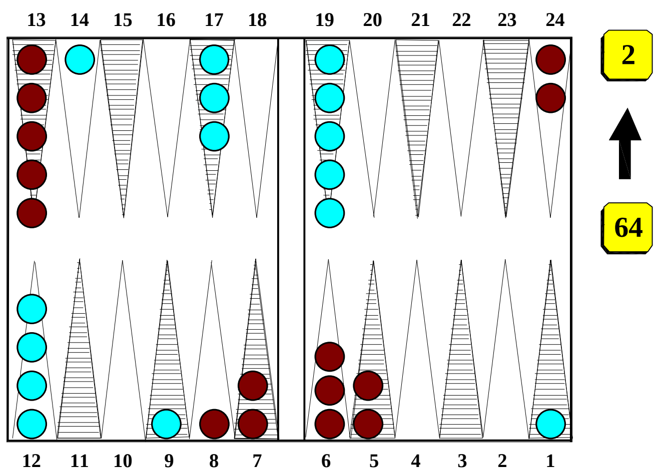
Table 2a Dead $(x = 0 \bullet 0)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	-0·500	-0·500	-0·500	-0·500	-0·500
cubeless	***1·25***	-0·500	-0·500	-0·500	-0·500	-0·500
loss	***1·50***	-0·500	-0·500	-0·500	-0·500	-0·500
value	***1·75***	-0·500	-0·500	-0·500	-0·500	-0·500
L	***2·00***	-0·500	-0·500	-0·500	-0·500	-0·500

Table 2b Live $(x = 1 \bullet 0)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	-0·600	-0·591	-0·583	-0·577	-0·571
cubeless	***1·25***	-0·636	-0·625	-0·615	-0·607	-0·600
loss	***1·50***	-0·667	-0·654	-0·643	-0·633	-0·625
value	***1·75***	-0·692	-0·679	-0·667	-0·656	-0·647
L	***2·00***	-0·714	-0·700	-0·688	-0·676	-0·667

Table 2c Normal $(x = \frac{2}{3})$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	-0·571	-0·565	-0·559	-0·554	-0·550
cubeless	***1·25***	-0·597	-0·588	-0·581	-0·575	-0·570
loss	***1·50***	-0·618	-0·608	-0·600	-0·593	-0·587
value	***1·75***	-0·635	-0·625	-0·616	-0·609	-0·602
L	***2·00***	-0·650	-0·640	-0·630	-0·622	-0·615

Example

Consider the following position, from the 12th game of the semi-finals match between Nack Ballard and Mike Senkiewicz at the Reno Masters in 1986. Senkiewicz, trailing 9-20 in this 23-point match, gave an initial double, which Ballard passed. Bill Robertie, analysing this match in his book *Reno Quiz*, evaluates the pass as correct at this match score. What would the correct cube action be in a money game?



Should White Take?

Using Robertie's cubeless rollout figures:

Black wins single-game:	47%
Black wins gammon:	17%
Black wins backgammon:	1%
Black's total wins:	65%
White wins single-game	31%
White wins gammon:	4%
White's total wins:	35%

Black's cubeless equity: 0·450 ppg

Considering White's cube action,

$$L = \frac{(47 + 17 \times 2 + 1 \times 3)}{(47 + 17 + 1)} = \underline{1 \cdot 292} \quad \text{and} \qquad W = \frac{(31 + 4 \times 2)}{(31 + 4)} = \underline{1 \cdot 114}$$

1. <u>Dead-Cube</u> (x = 0·0)

From equations (1) and (4):

$$TP_{dead} = \frac{(1 \cdot 292 - 0 \cdot 5)}{(1 \cdot 292 + 1 \cdot 114)} = \underline{0 \cdot 3292} \;\text{ and }\; E_{take} = 0 \cdot 3292 \times (1 \cdot 292 + 1 \cdot 114) - 1 \cdot 292 = -\underline{0 \cdot 500} \;\text{ clearly}$$

2. Live-Cube ($x = 1{\cdot}0$)

From equations (2) and (4):

$$TP_{live} = \frac{(1 \bullet 292 - 0 \bullet 5)}{(1 \bullet 292 + 1 \bullet 114 + 0 \bullet 5)} = \underline{0 \bullet 2725} \text{ and } E_{take} = 0 \bullet 2725 \times (1 \bullet 292 + 1 \bullet 114) - 1 \bullet 292 = -\underline{0 \bullet 636}$$

3. Normal-Cube ($x = \tfrac{2}{3}$)

From equations (3) and (4):

$$TP_{2/3} = \frac{(1 \bullet 292 - 0 \bullet 5)}{(1 \bullet 292 + 1 \bullet 114 + 0 \bullet 333)} = \underline{0 \bullet 2892} \text{ and } E_{take} = 0 \bullet 2892 \times (1 \bullet 292 + 1 \bullet 114) - 1 \bullet 292 = -\underline{0 \bullet 596}$$

In the actual position, White, with 35% winning chances, can take for money, regardless of the cube model considered.

Other Cube Action Decisions

So far, only take-points have been considered. There are many other doubling decisions to consider – when to redouble, when to beaver, etc. Correct cube-action can be established by comparing the resultant equities from the alternative cube positions – owned (E_O), unavailable (E_U), and centred (E_C):

$$E_O = C_V \left[p(W + L + 0 \bullet 5x) - L \right] \qquad \text{…equation (5)}$$

$$E_U = C_V \left[p(W + L + 0 \bullet 5x) - L - 0 \bullet 5x \right] \qquad \text{…equation (6)}$$

$$E_C = \frac{4C_V}{(4 - x)} \left[p(W + L + 0 \bullet 5x) - L - 0 \bullet 25x \right] \qquad \text{…equation (7)}$$

where C_V = cube-value (i.e., the stake-level)

p = cubeless winning probability

Note that equation (7) is not applicable if the *Jacoby Rule* is in operation.

From manipulation of the above equations, the following table of formulae, covering the full range of cube-actions in money games, has been derived. Notice two particularly interesting features from this table:

1. In the *live-cube model*, when gammons and backgammons are active, it is never correct to double, as positions strong enough to double are also *too good* to double! This is understandable because the complete lack of volatility protects the *double-out.*

2. Assuming the *Jacoby Rule* is not in operation, then initial double-points are always lower than redouble-points. When the cube is *dead* or *live*, they coincide, but diverge

for intermediate values of cube-life. Maximum divergence occurs when x is about 0·57, and typically ranges between 2.00% ($W = 2$, $L = 2$) and 3.75% ($W = 1$, $L = 1$).

Cube Action Formulae

Cube Parameter	Dead Cube $(x = 0 \cdot 0)$	Live Cube $(x = 1 \cdot 0)$	General Case $(x \text{ varies})$
Take-point, TP	$= \frac{(L-0\cdot5)}{(W+L)}$	$= \frac{(L-0\cdot5)}{(W+L+0\cdot5)}$	$= \frac{(L-0\cdot5)}{(W+L+0\cdot5x)}$
Beaver-point, BP	$= \frac{L}{(W+L)}$	$= \frac{L}{(W+L+0\cdot5)}$	$= \frac{L}{(W+L+0\cdot5x)}$
Racoon-point, RP	$= \frac{L}{(W+L)}$	$= \frac{(L+0\cdot5)}{(W+L+0\cdot5)}$	$= \frac{(L+0\cdot5x)}{(W+L+0\cdot5x)}$
Initial double-point, ID (no Jacoby)	$= \frac{L}{(W+L)}$	$= \frac{(L+1)}{(W+L+0\cdot5)}$	$= \frac{\left(L+\left(\frac{3-x}{2-x}\right)\frac{x}{2}\right)}{(W+L+0\bullet5x)}$
Initial double-point, ID_1 (Jacoby—no beavers)	$= \frac{(L-0\cdot5)}{(W+L-1)}$	$= \frac{(L+1)}{(W+L+0\cdot5)}$	$= \frac{k_1\left(L+\left(\frac{3-x}{2-x}\right)\frac{x}{2}\right)}{(W+L+0\bullet5x)}$ where $k_1 = \frac{(W+L)(L-0\cdot5(1-x))}{L(W+L-(1-x))}$
Initial double-point, ID_2 (Jacoby with beavers)	$= \frac{(L-0\cdot25)}{(W+L-0\cdot5)}$ $\not< \frac{(L-0\cdot5)}{(W+L-1)}$	$= \frac{(L+1)}{(W+L+0\cdot5)}$	$= \frac{k_2\left(L+\left(\frac{3-x}{2-x}\right)\frac{x}{2}\right)}{(W+L+0\bullet5x)}$ where $k_2 = \frac{(W+L)(L-0\cdot25(1-x))}{L(W+L-0\bullet5(1-x))} \not< k_1$
Redouble-point, RD	$= \frac{L}{(W+L)}$	$= \frac{(L+1)}{(W+L+0\cdot5)}$	$= \frac{(L+x)}{(W+L+0\cdot5x)}$
Cash-point, CP	$= \frac{(L+0\cdot5)}{(W+L)}$	$= \frac{(L+1)}{(W+L+0\cdot5)}$	$= \frac{(L+0\cdot5+0\cdot5x)}{(W+L+0\cdot5x)}$
Too good point, TG	$= \frac{(L+1)}{(W+L)}$	$= \frac{(L+1)}{(W+L+0\cdot5)}$	$= \frac{(L+1)}{(W+L+0\cdot5x)}$

where $W =$ Average cubeless value of games ultimately won

$L =$ Average cubeless value of games ultimately lost

$x =$ Cube life index (0·0 for dead cube, 1·0 for live cube)

$k_1 =$ Jacoby factor (no beavers)

$k_2 =$ Jacoby factor (with beavers)

Appendix 1: Miscellaneous Equity Relationships

The various equities for the different cube positions may be expressed in terms of the cube-life index (x), cubeless probability of winning (p), and cubeless equity (E) as follows:

Cubeless Equity $\quad E = p(W + L) - L$

Cube-owned Equity $\quad E_O = C_V\left[E + 0\cdot 5x\, p\right]$

Cube-unavailable Equity $\quad E_U = C_V\left[E - 0\bullet 5x(1 - p)\right]$

Cube-centred Equity $\quad E_C = \dfrac{4C_V}{(4 - x)}\left[E + 0\bullet 5x(p - 0\bullet 5)\right]$

The cube-centred equity may also be expressed in terms of the cube-owned and cube-unavailable equities (with their respective C_V values set at unity) as follows:

$$E_C \;=\; \frac{4}{(4 - x)}\left(E_O - 0\bullet 25x\right) \;=\; \frac{4}{(4 - x)}\left(E_U + 0\bullet 25x\right) \;=\; \frac{2}{(4 - x)}\left(E_O + E_U\right)$$

Note that the cube-centred equity formulae given above are not applicable if the *Jacoby Rule* is in operation. The cube-owned and cube-unavailable equities corresponding to the various cube-action points are shown by the following table:

Cube Parameter	**Cube-owned Equity** E_O	**Cube-unavailable Equity** E_U
Take-point	$-0\cdot 5C_V$	$-0\bullet 5(1 + x)C_V$
Beaver-point	0	$-0\cdot 5xC_V$
Racoon-point	$+0\cdot 333xC_V$	0
Initial double-point (no Jacoby)	$+\dfrac{x}{2}\dfrac{(3 - x)}{(2 - x)}C_V$	$+\dfrac{x}{(4 - 2x)}C_V$
Initial double-point (Jacoby—no beavers)	$C_V\left[k_1 \frac{x}{2}\left(\frac{3-x}{2-x}\right) + L(k_1 - 1)\right]$	$C_V\left[\frac{x}{2}\left(k_1\left(\frac{3-x}{2-x}\right) - 1\right) + L(k_1 - 1)\right]$
Initial double-point (Jacoby with beavers)	$C_V\left[k_2 \frac{x}{2}\left(\frac{3-x}{2-x}\right) + L(k_2 - 1)\right]$	$C_V\left[\frac{x}{2}\left(k_2\left(\frac{3-x}{2-x}\right) - 1\right) + L(k_2 - 1)\right]$
Redouble-point	$+xC_V$	$+0\cdot 5xC_V$
Cash-point	$+0\bullet 5(1 + x)C_V$	$+0\cdot 5C_V$
Too good point	$+C_V$	$+(1 - 0\bullet 5x)C_V$

Note that the above equities are independent of *W* and *L* apart for the initial double equities with the *Jacoby Rule* in operation. Also note that the cube-unavailable equity required for a redouble is the cube-life index (*x*) multiplied by the stake of the redoubled cube (C_V). Using ⅔ as a *normal* value for *x*, the required equities after doubling to 2 are 0·667 and 0·500, for redoubles and initial doubles (no *Jacoby*) respectively. These values are fairly consistent with typical limiting values obtained from hand rollouts (generally minimum redoubles are between 0·6 and 0·7 ppg, and between 0·4 and 0·6 ppg for initial doubles). Consequently, ⅔ would appear to be a good estimate of the cube-life index.

Appendix 2: Refined General Model

A more rigorous analysis may be performed by considering different *cube-life indices* for both sides, which is what normally happens in practice. Let x_1 and x_2 be the *cube-life indices* for the player making the cube-action decision, and his opponent, respectively. Following a similar analysis as before, the equities from the alternative cube positions, owned (E_O), unavailable (E_U), and centred (E_C), were derived:

$$E_O = C_V \left[p\left(W + L + 0 \bullet 5x_1\right) - L \right]$$ …equation (8)

$$E_U = C_V \left[p\left(W + L + 0 \bullet 5x_2\right) - L - 0 \bullet 5x_2 \right]$$ …equation (9)

$$E_C = \frac{C_V \left[2p\left(W + L + 0 \bullet 5x_2\right) - Q_x(L+1) - \left(L + 0 \bullet 5x_2 - 1\right)\right]}{\left[Q_x(L+1) - \left(L + 0 \bullet 5x_2 - 1\right)\right]}$$ …equation (10)

where $Q_x = \dfrac{\left(W + L + 0 \bullet 5x_2\right)}{\left(W + L + 0 \bullet 5x_1\right)}$

C_V = cube-value (i.e., the stake-level)

p = cubeless winning chances

The cube-centred-equity (E_C), can also be estimated from the simpler, but more approximate, expression:

$$E_C \approx 4C_V \frac{\left(x_1 E_O + x_2 E_U\right)}{\left[4\left(x_1 + x_2\right) - 2x_1 x_2\right]}$$ …equation (11)

where E_O and E_U are calculated from equations (8) and (9) above, with C_V equal to 1·0 in both cases. Note that equations (10) and (11) are not applicable if the *Jacoby Rule* is in operation.

From manipulation of the above equations, the following table of cube-action formulae, allowing for different *cube-life indices* for both sides, has been derived:

Cube Action Formulae (Refined General Model)

Cube Parameter	Refined General Model Formula $\left(x_1 \text{ and } x_2 \text{ vary}\right)$
Take-point	$TP = \dfrac{(L - 0\cdot 5)}{(W + L + 0\cdot 5x_1)}$
Beaver-point	$BP = \dfrac{L}{(W + L + 0\cdot 5x_1)}$
Racoon-point	$RP = \dfrac{(L + 0\cdot 5x_2)}{(W + L + 0\cdot 5x_2)}$
Initial-double point (no Jacoby)	$ID = \dfrac{(L + 0\cdot 5x_2 - 0\cdot 5 + G_x)}{(W + L + 0\cdot 5x_2)}$ where $G_x = \dfrac{0 \bullet 5\left(W + L + 0 \bullet 5x_1\right)}{(L+1)\left(W + L + 0 \bullet 5x_2\right) - \left(L + 0 \bullet 5x_2\right)\left(W + L + 0 \bullet 5x_1\right)}$ alternatively, $ID \approx \dfrac{\left(L + H_x x_2\right)}{\left(W + L + x_2 - 0 \bullet 5x_1\right)}$ where $H_x = \dfrac{\left(3 - x_2\right)}{\left(4 - 2x_2\right)}$
Initial double-point (Jacoby—no beavers)	$ID_1 = k_1 ID$ where $k_1 = \dfrac{(W + L)\left(L - 0\cdot 5\left(1 - x_C\right)\right)}{L\left(W + L - \left(1 - x_C\right)\right)}$ and $x_C = \dfrac{x_1(W-1) + x_2(L-1)}{(W + L - 2)}$
Initial double-point (Jacoby with Beavers)	$ID_2 = k_2 ID$ where $k_2 = \dfrac{(W + L)\left(L - 0\cdot 25\left(1 - x_C\right)\right)}{L\left(W + L - 0 \bullet 5\left(1 - x_C\right)\right)} \not< k_1$
Redouble-point	$RD = \dfrac{(L + x_2)}{(W + L + x_2 - 0\cdot 5x_1)}$
Cash-point	$CP = \dfrac{(L + 0\cdot 5x_2 + 0\cdot 5)}{(W + L + 0\cdot 5x_2)}$
Too Good point	$TG = \dfrac{(L+1)}{(W + L + 0\cdot 5x_1)}$

Appendix 3: Jacoby Rule Considerations

The *Jacoby Rule* suppresses gammons and backgammons until the cube is first turned. Consequently, initial double points can be markedly different from the corresponding points where gammons and backgammons are active with a centred-cube. To help us to understand this general relationship, consider how the number of market-losing sequences required for an initial double varies:

$$n = \frac{N}{\left[1 + \frac{\Delta E_P}{\Delta E_N}\right]} \qquad \text{…equation (12)}$$

where n is the minimum number of market-losing sequences required for a double, N is the (population size, commonly 1296, occasionally 36), ΔE_P is the average *favourable* equity swing from the market-losing sequences, and ΔE_N is the average *adverse* equity swing from the non-market-losing sequences (always of positive value in this equation).

With regard to the market losing sequences, the average equity with the cube turned is the same value regardless of *Jacoby* considerations as gammons are activated in either case. If the cube is not turned, however, the average equities can be different dependent on whether gammons are active or not (i.e., the position might become too good rather than just a cash). Accordingly, the positive equity swings can often be greater (never less) when the *Jacoby Rule* is in operation than would be the case otherwise. Inspection of equation (13) shows that this has the net effect of ***reducing*** the number of market-losing sequences (and thus winning chances) required for an initial double.

With regard to the non-market-losing sequences, again there is no difference in the average equities with the cube turned. If the cube remained centred, you cannot become too good (otherwise you would have lost your market), but your opponent might! Consequently, the negative equity swings can often be greater (never less) when the *Jacoby Rule* is operation, with the net effect of ***increasing*** the number of market-losers required for an initial double.

What is the overall effect of the *Jacoby Rule* on initial doubling strategy? In general terms, you should be more ***aggressive*** with the cube than normal, when you are likely to win a greater proportion of gammons than your opponent (i.e., *W* exceeds *L*), and more ***conservative*** otherwise. Interestingly and significantly, where an aggressive policy is indicated, the prospect of the double being beavered (correctly) has the effect of curbing that aggression.

The degree of modification to initial cube action policy from *normal* is directly related to the tendency of a position to suddenly become *too good,* for either side. This tendency is roughly proportional to the volatility of the position. In the *live cube model* (zero volatility), the initial double point is unaffected by *Jacoby* considerations: it still coincides with the cash point, as the margin of market-loss is non-existent. In the d*ead-cube model* (maximum volatility) however, this effect is at its most extreme: complete market loss occurs for the winning side, and if he has any gammons his position will become too good.

How, then can these effects and tendencies be incorporated into a general cube model? We can make a start by introducing the following general relationship:

$$ID_J = k\ ID \qquad \text{…equation (13)}$$

where ID_J is the initial double-point with the *Jacoby Rule* in operation, *k* is a relational parameter, as yet undefined, which we'll call the *Jacoby factor*, and *ID* is the *normal* initial double-point. To define *k*, the following factors must be taken into account:

1. The limiting values of *k* at the extremities of the *volatility spectrum* (dead-cube and live-cube models). These can be determined fairly readily.

2. Whether beavers are allowed or not.

3. The method by which intermediate volatility, and thus cube-life, is modelled, i.e., the *basic* general model (*x*), or the *refined* general model (x_1 and x_2).

Initially, the simpler *basic* general model will be considered, for the two beaver-cases, before the more complicated *refined* general model is tackled.

Jacoby—no Beavers

The general relationship given in equation (13) above can be redefined for this specific case as follows:

$$ID_1 = k_1 ID \qquad \text{…equation (14)}$$

where ID_1 is the initial double-point with the *Jacoby Rule* in operation and no beavers allowed, k_1 is the Jacoby factor (no beavers), and *ID* is the *normal* initial double-point. When the cube is live ($x = 1 \cdot 0$), k_1 is of unit value, as both initial double-points coincide with the cash-point. When the cube is dead, however, k_1 has the following non-trivial value:

$$k_{1\,dead} = \frac{(W+L)(L-0\bullet5)}{L(W+L-1)} \qquad \text{…equation (15)}$$

The following table shows how k_{1dead} varies with *W* and *L*.

Jacoby Factor k_1 ($x = 0\bullet0$)		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	1·000	0·900	0·833	0·786	0·750
cubeless	***1·25***	1·080	1·000	0·943	0·900	0·867
loss	***1·50***	1·111	1·048	1·000	0·963	0·933
value	***1·75***	1·122	1·071	1·032	1·000	0·974
L	***2·00***	1·125	1·083	1·050	1·023	1·000

Notice some important features from this table:

1. For all positions where *W* is equal to *L*, there is no difference in initial doubling strategy from *normal*.

2. For all positions where *W* exceeds *L*, initial doubling strategy with the *Jacoby Rule* in operation is more ***aggressive*** than *normal*, as it is beneficial to activate gammons. Note that in these positions it is correct to double even when your equity is negative!

3. For all positions where *L* exceeds *W*, initial doubling strategy with the *Jacoby Rule* in operation is more ***conservative*** than *normal*, as it is disadvantageous to activate gammons. Note that in these positions, greater equity is required for an initial double than a redouble (Latto's Paradox).

Having defined k_1 at the extremities of the *volatility spectrum*, it remains to formulate an algorithm for all intermediate values. The simplest expression which satisfies our limited criteria is given below:

$$k_1 = \frac{(W+L)\left(L - 0\cdot 5(1-x)\right)}{L\left(W+L-(1-x)\right)}$$ …equation (16)

This formula assumes a roughly linear relationship between the Jacoby factor and the cube-life index, which certainly seems reasonable. Besides, if the relationship is not linear, in which direction should it curve, and what should its curvature be? In the absence of any greater understanding, a more elaborate algorithm would serve no useful purpose. Even if a more accurate algorithm was available, it is unlikely that the greater precision afforded would be significant.

The following table shows how k_1 varies with *W* and *L*, for a *typical* position ($x = ⅔$).

Jacoby Factor k_1 $(x = ⅔)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	1·000	0·978	0·962	0·948	0·938
cubeless	***1·25***	1·017	1·000	0·986	0·975	0·966
loss	***1·50***	1·026	1·011	1·000	0·990	0·982
value	***1·75***	1·030	1·018	1·008	1·000	0·993
L	***2·00***	1·031	1·021	1·013	1·006	1·000

Note that the maximum difference between the Jacoby and non-Jacoby values is about 6%, and the greater differences occur when *W* exceeds *L*.

Jacoby with Beavers

The general relationship given in equation (13) above can be redefined for this specific case as follows:

$$ID_2 = k_2 ID$$ …equation (17)

where ID_2 is the initial double-point with the *Jacoby Rule* in operation with beavers allowed, k_2 is the Jacoby factor (with beavers), and *ID* is the *normal* initial double-point. When the cube is live ($x = 1 \cdot 0$), k_2 is of unit value, as both initial double-points coincide with the cash-point. When the cube is dead, however, k_2 must be the greater of either k_{1dead} (when beavering is wrong) or the following expression:

$$k_{2\,dead} = \frac{(W + L)(L - 0 \cdot 25)}{L(W + L - 0 \bullet 5)} \not< k_{1\,dead} \qquad \text{…equation (18)}$$

The following table shows how k_{2dead} varies with *W* and *L*.

Jacoby Factor k_2 ($x = 0 \bullet 0$)		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	1·000	0·964	0·938	0·917	0·900
cubeless	***1·25***	1·080	1·000	0·978	0·960	0·945
loss	***1·50***	1·111	1·048	1·000	0·985	0·972
value	***1·75***	1·122	1·071	1·032	1·000	0·989
L	***2·00***	1·125	1·083	1·050	1·023	1·000

Notice some important features from this table:

1. For all positions where *W* is equal to *L*, there is no difference in initial doubling strategy from *normal*.

2. For all positions where *W* exceeds *L*, initial doubling strategy with the *Jacoby Rule* in operation is more ***aggressive*** than *normal*, as it is beneficial to activate gammons. Note that in these positions it is correct to double even when your equity is negative, and it is correct for your opponent to beaver! These are pure *Kauder paradox* positions.

3. For all positions where *L* exceeds *W*, initial doubling strategy with the *Jacoby Rule* in operation is more ***conservative*** than *normal*, as it is disadvantageous to activate gammons. Note that in these positions, greater equity is required for an initial double than a redouble (Latto's Paradox). Also note that as beavering is incorrect in these positions, there is no difference from the *no-beavers* case.

Having defined k_2 at the extremities of the *volatility spectrum*, it remains to formulate an algorithm for all intermediate values. The simplest expression which satisfies our limited criteria is given below:

$$k_2 = \frac{(W + L)\big(L - 0 \cdot 25(1 - x)\big)}{L\big(W + L - 0 \bullet 5(1 - x)\big)} \not< k_1 \qquad \text{…equation (19)}$$

This formula assumes a roughly linear relationship between the Jacoby factor and the cube-life index, as does equation (16) for the *no beavers* case. As discussed previously, this assumption is certainly reasonable, and it is unlikely that the greater precision afforded by a more accurate algorithm would be significant, even were it available.

The following table shows how k_2 varies with *W* and *L*, for a *normal* position ($x = 2/3$).

Jacoby Factor k_2 ($x = 2/3$)		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	1·000	0·990	0·982	0·976	0·971
cubeless	***1·25***	1·017	1·000	0·994	0·988	0·984
loss	***1·50***	1·026	1·011	1·000	0·995	0·992
value	***1·75***	1·030	1·018	1·008	1·000	0·997
L	***2·00***	1·031	1·021	1·013	1·006	1·000

Note that the maximum difference between the Jacoby and non-Jacoby values is about 3% (compared to 6% with the *no-beavers* case), and the differences are roughly proportional to the difference between *W* and *L* (i.e., $W - L$).

Jacoby and the Refined General Model

When separate cube-life indices are considered for the player making the cube-action decision and his opponent (x_1 and x_2 respectively), the algorithms we require for k_1 and k_2 are slightly more difficult to formulate, as they must depend on both x_1 and x_2. The simplest approach is to utilise the previously derived expressions for k_1 and k_2 by considering a composite value of x_1 and x_2 (x_C) which satisfies the following limiting criteria:

1. When $x_1 = x_2$ then $x_C = x_1 = x_2$. Clearly when the *refined* general model simplifies to the *basic* general model, the *basic* model's expressions for k_1 and k_2 must still hold good.

2. When $W = 1$ then $x_C = x_2$. When you cannot win any gammons or backgammons, only your opponent can become *too good*. Consequently the initial double-point cannot be dependent on your volatility, and must, by elimination, be dependent on his.

3. When $L = 1$ then $x_C = x_1$. When your opponent cannot win any gammons or backgammons, only you can become *too good*. Consequently the initial double-point cannot be dependent on his volatility, and must, by elimination, be dependent on yours.

The simplest expression which satisfies the above criteria is given below:

$$x_C = \frac{x_1(W-1) + x_2(L-1)}{(W+L-2)} \qquad \text{…equation (20)}$$

For the ***Jacoby—no Beavers*** case, initial double points can be calculated from equation (14) and the suitably revised version of equation (16), both given below for clarity:

$$ID_1 = k_1 ID \qquad \text{…equation (14)}$$

$$k_1 = \frac{(W+L)\left(L-0\cdot5\left(1-x_C\right)\right)}{L\left(W+L-\left(1-x_C\right)\right)}$$ …equation (21)

where *ID* is calculated from the relevant equation in Appendix 1.

For the ***Jacoby with Beavers*** case, initial double points can be calculated from equation (17) and the suitably revised version of equation (19), both given below for clarity:

$$ID_2 = k_2 ID$$ …equation (17)

$$k_2 = \frac{(W+L)\left(L-0\cdot25\left(1-x_C\right)\right)}{L\left(W+L-0\bullet5\left(1-x_C\right)\right)} \not< k_1$$ …equation (22)

As before, this analysis assumes roughly linear relationships between the Jacoby factors and the cube-life indices. This assumption is certainly reasonable, and it is unlikely that the greater precision afforded by a more accurate algorithm would be significant, even were it available.

Cube-centred Equities

No simple formula is available to calculate cube-centred equities, but we do know four points where the cubeless winning chances and corresponding equities are known. These are given below, in order of increasing probability (and equity):

Point 1: The opponent's cash-point, where

$$p_1 = \frac{\left(L+0\bullet5x_2-1\right)}{\left(W+L+0\bullet5x_2\right)} \quad \text{and} \quad E_{C\,1} = -1 \text{ ppg}$$

If using the basic general model, substitute x for x_2.

Point 2: The opponent's initial double-point, where

$$p_2 = 1-ID_1 \text{ or } 1-ID_2 \quad \text{and} \quad E_{C\,2} = 2\left[p_2\left(W+L+0\bullet5x_1\right)-L\right]$$

If using the basic general model, substitute x for x_1.

Point 3: Your initial double-point, where

$$p_3 = ID_1 \text{ or } ID_2 \quad \text{and} \quad E_{C\,3} = 2\left[p_3\left(W+L+0\bullet5x_2\right)-L-0\bullet5x_2\right]$$

If using the basic general model, substitute x for x_2.

Point 4: Your cash-point, where

$$p_4 = \frac{(L+1)}{(W+L+0 \bullet 5x_1)} \quad \text{and} \quad E_{C\,4} = +1 \text{ ppg}$$

If using the basic general model, substitute x for x_1.

The cube-centred equity can then be estimated from a known cubeless probability by interpolation between the two known probabilities directly above and below it by using the following general formula:

$$E_C = E_n + (E_{n+1} - E_n)\frac{(p_{n+1} - p)}{(p_{n+1} - p_n)}$$

Similarly, the cubeless probability can be estimated from a known cube-centred equity by interpolation between the two known cube-centred equities directly above and below it by using the following modified version of the same general formula:

$$p = p_n + (p_{n+1} - p_n)\frac{(E_{n+1} - E_C)}{(E_{n+1} - E_n)}$$

A more elaborate analysis could be carried out by fitting a polynomial curve through the four points, but the greater sophistication is unlikely to improve accuracy significantly.

Kauder Paradox Positions

Kauder paradox positions occur when it is correct to give an initial double, yet the opponent should beaver. They arise because the *Jacoby Rule* occasionally allows the doubler to minimise his losses. The mathematical condition for a *Kauder paradox* is given below:

$$RP \geq p \geq ID_2$$

where RP is the racoon-point, p is the cubeless winning probability, and ID_2 is the initial double-point. Similarly, the *Kauder paradox window* (*KPW)*, can be expressed as:

$$KPW = RP - ID_2$$

If *KPW* is negative, then a *Kauder paradox* cannot occur. From the *basic* general cube-model, the following more detailed expression for the *Kauder paradox window* was readily established.

$$KPW = \frac{\left[L + 0 \bullet 5x - k_2\left(L + \left(\frac{3-x}{2-x}\right)\frac{x}{2}\right)\right]}{(W + L + 0 \bullet 5x)} \qquad \text{…equation (23)}$$

Clearly, *Kauder paradoxes* are more likely to occur when the position is volatile (and thus x is small). The following table shows the limiting x values, calculated from equation (23), above which a *Kauder paradox* cannot occur, against values of W and L:

Kauder Paradox Limiting x values		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·000	0·124	0·198	0·248	0·285
cubeless	***1·25***	none	0·000	0·099	0·164	0·210
loss	***1·50***	none	none	0·000	0·082	0·140
value	***1·75***	none	none	none	0·000	0·070

L	***2·00***	none	none	none	none	0·000

From inspection of the above values, the following approximate expressions for the limiting x values (x_{KP}) was established:

$$x_{KP} \approx 0 \bullet 59 - 0 \bullet 01\frac{W}{L} - 0 \bullet 58\frac{L}{W}$$ …equation (24)

$$x_{KP} \approx 0 \bullet 58 - 0 \bullet 58\frac{L}{W}$$ …equation (24A)

Notice how both the volatility and favourable gammon rate must be high for a *Kauder paradox* to be possible, which is not too surprising.

Latto's Paradox Positions

Latto's paradox positions occur when a redouble is correct but an initial double is not. They arise because the *Jacoby Rule* occasionally allows the doubler to maximise his winnings by avoiding gammon losses. The mathematical condition for a *Latto's paradox* is given below:

$$ID_1 \geq p \geq RD$$

where RD is the redouble-point, p is the cubeless winning probability, and ID_1 is the initial double-point (or ID_2 which has the same value here). Similarly, the *Latto's paradox window* (*LPW*), can be expressed as:

$$LPW = ID_1 - RD$$

If LPW is negative, then a *Latto's paradox* cannot occur. From the *basic* general cube-model, the following more detailed expression for the *Latto's paradox window* was readily established.

$$LPW = \frac{\left[k_1\left(L + \left(\tfrac{3-x}{2-x}\right)\tfrac{x}{2}\right) - L - x\right]}{(W + L + 0 \bullet 5x)}$$ …equation (25)

Note that k_1 can be substituted by k_2 which has the same value here.

Latto's paradoxes, just like their *Kauder* counterparts, are more likely to occur when the position is volatile (and thus x is small). The following table shows the limiting x values, calculated from equation (25), above which a *Latto's paradox* cannot occur, against values of W and L:

Latto's Paradox Limiting x values		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·000	none	none	none	none
cubeless	***1·25***	0·322	0·000	none	none	none
loss	***1·50***	0·484	0·248	0·000	none	none
value	***1·75***	0·584	0·396	0·200	0·000	none

L	***2·00***	0·652	0·496	0·333	0·167	0·000

From inspection of the above values, the following approximate expressions for the limiting *x* values (x_{LP}) was established:

$$x_{LP} \approx 1{\cdot}6 - 1{\cdot}5\frac{W}{L} - 0{\cdot}1\frac{L}{W} \qquad \text{…equation (26)}$$

$$x_{LP} \approx 1 \bullet 3 - 1 \bullet 3\frac{W}{L} \qquad \text{…equation (26A)}$$

Notice that *Latto's paradoxes*, unlike their *Kauder* counterparts, don't need particularly high volatility for them to be possible. With very high unfavourable gammon rates, they can occur under *normal* cube-life conditions. This is certainly a surprising result, to me anyway. What are the reasons that they appear to be rare? Here are a few possibilities:

1. In the vast majority of positions, the player with the favourable gammons reaches an initial doubling position first. This is because both players start as roughly equal favourites with roughly equal gammon chances. The player who gets the better of the early game has usually done so by hitting shots, creating a blockade, escaped his back men, or by gaining an edge in the race. These variations rarely lead to unfavourable gammons, quite the contrary.

2. *Latto's paradox* positions usually arise after a significant change in fortune, e.g., leaving multiple shots to the opponent's deep anchor or back game position. Remember, your opponent needs to become *too good* after your non-market losing sequences for the *Jacoby Rule* to have any effect on *normal* doubling policy. Normally, your opponent would have doubled you before this change in fortune happens. Moreover, the *Jacoby Rule* encourages his aggressive initial cube-action.

3. All too often, many players don't recognise the conditions that call for conservative initial-cube action, and incorrectly give premature doubles, which would otherwise be reasonable if the *Jacoby Rule* was not in operation.

4. Many players think the *Jacoby Rule* in general calls for aggressive initial-cube action. They may or may not know that the converse is also possible. Whichever
way you look at it, the likelihood is that doubles will often occur sooner than might be technically correct.

Appendix 4: Miscellaneous Equity Tables

Take-point $(x = \frac{2}{3})$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	-0·571	-0·565	-0·559	-0·554	-0·550
cubeless	***1·25***	-0·597	-0·588	-0·581	-0·575	-0·570
loss	***1·50***	-0·618	-0·608	-0·600	-0·593	-0·587
value	***1·75***	-0·635	-0·625	-0·616	-0·609	-0·602
L	***2·00***	-0·650	-0·640	-0·630	-0·622	-0·615

Table A1: Cubeless Take Equities

$(x = \frac{2}{3})$

Beaver-point $(x = \frac{2}{3})$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	-0·143	-0·129	-0·118	-0·108	-0·100
cubeless	***1·25***	-0·161	-0·147	-0·135	-0·125	-0·116
loss	***1·50***	-0·176	-0·162	-0·150	-0·140	-0·130
value	***1·75***	-0·189	-0·175	-0·163	-0·152	-0·143
L	***2·00***	-0·200	-0·186	-0·174	-0·163	-0·154

Table A2: Cubeless Beaver Equities

$(x = \frac{2}{3})$

Racoon-point $(x = \frac{2}{3})$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·143	0·161	0·176	0·189	0·200
cubeless	***1·25***	0·129	0·147	0·162	0·175	0·186
loss	***1·50***	0·118	0·135	0·150	0·163	0·174
value	***1·75***	0·108	0·125	0·140	0·152	0·163
L	***2·00***	0·100	0·116	0·130	0·143	0·154

Table A3: Cubeless Racoon Equities

$(x = \frac{2}{3})$

Init. double, ID $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·357	0·379	0·397	0·412	0·425
cubeless	***1·25***	0·347	0·368	0·385	0·400	0·413
loss	***1·50***	0·338	0·358	0·375	0·390	0·402
value	***1·75***	0·331	0·350	0·366	0·380	0·393
L	***2·00***	0·325	0·343	0·359	0·372	0·385

Table A4: Cubeless Initial-Double Equities (no Jacoby)

$(x = 2/3)$

Init. double, ID_1 $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·357	0·349	0·343	0·339	0·336
cubeless	***1·25***	0·375	0·368	0·363	0·359	0·356
loss	***1·50***	0·385	0·379	0·375	0·372	0·369
value	***1·75***	0·393	0·388	0·384	0·380	0·378
L	***2·00***	0·398	0·393	0·390	0·387	0·385

Table A5: Cubeless Initial-Double Equities (Jacoby—no beavers)

$(x = 2/3)$

Init. double, ID_2 $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·357	0·365	0·372	0·378	0·383
cubeless	***1·25***	0·375	0·368	0·375	0·381	0·386
loss	***1·50***	0·385	0·379	0·375	0·381	0·386
value	***1·75***	0·393	0·388	0·384	0·380	0·386
L	***2·00***	0·398	0·393	0·390	0·387	0·385

Table A6: Cubeless Initial-Double Equities (Jacoby with beavers)

$(x = 2/3)$

Redouble $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·429	0·452	0·471	0·486	0·500
cubeless	***1·25***	0·419	0·441	0·459	0·475	0·488
loss	***1·50***	0·412	0·432	0·450	0·465	0·478
value	***1·75***	0·405	0·425	0·442	0·457	0·469
L	***2·00***	0·400	0·419	0·435	0·449	0·462

Table A7: Cubeless Redouble Equities

$(x = 2/3)$

Cash $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·571	0·597	0·618	0·635	0·650
cubeless	***1·25***	0·565	0·588	0·608	0·625	0·640
loss	***1·50***	0·559	0·581	0·600	0·616	0·630
value	***1·75***	0·554	0·575	0·593	0·609	0·622
L	***2·00***	0·550	0·570	0·587	0·602	0·615

Table A8: Cubeless Cash Equities

$(x = 2/3)$

Too Good $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·714	0·742	0·765	0·784	0·800
cubeless	***1·25***	0·710	0·735	0·757	0·775	0·791
loss	***1·50***	0·706	0·730	0·750	0·767	0·783
value	***1·75***	0·703	0·725	0·744	0·761	0·776
L	***2·00***	0·700	0·721	0·739	0·755	0·769

Table A9: Cubeless Too Good Equities

$(x = 2/3)$

Init. double, ID $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·500	0·500	0·500	0·500	0·500
cubeless	***1·25***	0·500	0·500	0·500	0·500	0·500
loss	***1·50***	0·500	0·500	0·500	0·500	0·500
value	***1·75***	0·500	0·500	0·500	0·500	0·500
L	***2·00***	0·500	0·500	0·500	0·500	0·500

Table A10: Cube-Centred Initial-Double Equities (no Jacoby)

$(x = 2/3)$

Init. double, ID_1 $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·500	0·431	0·378	0·336	0·302
cubeless	***1·25***	0·564	0·500	0·449	0·408	0·374
loss	***1·50***	0·607	0·548	0·500	0·460	0·427
value	***1·75***	0·638	0·583	0·538	0·500	0·467
L	***2·00***	0·661	0·611	0·568	0·532	0·500

Table A11: Cube-Centred Initial-Double Equities (Jacoby—no beavers)

$(x = 2/3)$

Init. double, ID_2 $(x = 2/3)$		Average cubeless win value ***W***				
		1·00	***1·25***	***1·50***	***1·75***	***2·00***
Average	***1·00***	0·500	0·468	0·443	0·423	0·407
cubeless	***1·25***	0·564	0·500	0·476	0·457	0·441
loss	***1·50***	0·607	0·548	0·500	0·481	0·465
value	***1·75***	0·638	0·583	0·538	0·500	0·484
L	***2·00***	0·661	0·611	0·568	0·532	0·500

Table A12: Cube-Centred Initial-Double Equities (Jacoby with beavers)

$(x = 2/3)$

Appendix 5: Derivation of Live-Cube Take Point Formulae

1. Infinite Model — Constant gammon and backgammon-rates

Assumptions:

1. Infinite number of possible subsequent optimal redoubles.
2. Owner of cube, when he redoubles, is guaranteed to use it with perfect efficiency, at which point his opponent will have an optional pass/take.
3. Gammon and backgammon rates are constant.

Player A and player B play backgammon for money. Player A's average win value is W, and his average loss value is L. If player A owns the cube, the game is effectively played between the limits $p = 0$ (when A loses) and $p = CP$ (when A cashes), where p is player A's cubeless winning probability and CP is his cash-point (take-point for player B). If we define q as player A's effective winning probability owning the cube, then the following relationships can be established:

When $p = 0$, $q = 0$, and when $p = CP$, $q = 1$.

As the game is *continuous*, intermediate values may be found by linear interpolation, as follows:

$$q = \frac{p}{CP} \qquad \text{... (P1)}$$

Assume player A is doubled from the current stake-level of C_V to the new stake-level of $2C_V$. His take-point, in terms of effective winning probability owning the cube, may be established from the risk-reward ratio — he is risking $2C_V L - C_V$ points to gain $2C_V + C_V = 3C_V$ points. Although player A stands to lose L points at the new stake if he takes and loses, he only stands to win 1 point at the new stake when he takes and wins, because he cashes all the games he wins — his W is realised in another way, the ability to cash sooner. The effective take-point (TP_O) is given by the following expression:

$$TP_O = \frac{\left(2C_V L - C_V\right)}{\left(2C_V L - C_V + 3C_V\right)} = \frac{(L - 0\cdot 5)}{(L+1)} \qquad \text{... (P2)}$$

When $q = TP_O$, $p = TP$ (the take-point in terms of cubeless winning probability). Therefore, from equation (P1),

$$TP_O = \frac{TP}{CP} \qquad \text{... (P3)}$$

By substituting into equation (P2), we derive the following relationship:

$$\frac{TP}{CP} = \frac{(L - 0\cdot 5)}{(L+1)} \qquad \text{... (P4)}$$

A similar relationship can be derived from consideration of player B's effective take-point. This may be done by making the following substitutions in equation (P4):

Substitute TP by $1 - CP$ (player B's take-point)
Substitute CP by $1 - TP$ (player B's cash-point)
Substitute L by W (player B's mean win value)

$$\therefore \frac{(1-CP)}{(1-TP)} = \frac{(W-0\cdot5)}{(W+1)} \qquad \text{... (P5)}$$

Solving equations (P4) and (P5) simultaneously, gives us the expression for the *live-cube* take point:

$$TP = \frac{(L-0\cdot5)}{(L+W+0\cdot5)} \qquad \text{... (P6)}$$

2. Finite Model — Constant gammon and backgammon-rates

Assumptions:
1. Finite number of possible subsequent optimal redoubles.
2. Owner of cube, when he redoubles, is guaranteed to use it with perfect efficiency, at which point his opponent will have an optional pass/take.
3. Gammon and backgammon rates are constant.

Player A and player B play backgammon for money. Player A's average win value is *W*, and his average loss value is *L*. If player A owns the cube, the game is effectively played between the limits $p = 0$ (when A loses) and $p = CP$ (when A cashes), where *p* is player A's cubeless winning probability and *CP* is his cash-point (take-point for player B). If we define *q* as player A's effective winning probability owning the cube, then the following relationships can be established:

When $p = 0$, $q = 0$, and when $p = CP$, $q = 1$.

As the game is *continuous*, intermediate values may be found by linear interpolation, as follows:

$$q = \frac{p}{CP} \qquad \text{... (P1)}$$

Assume player A is doubled from the current stake-level of C_V to the new stake-level of $2C_V$.
His take-point, in terms of effective winning probability owning the cube, may be established from the risk-reward ratio — he is risking $2C_V L - C_V$ points to gain $2C_V + C_V = 3C_V$ points. Although player A stands to lose *L* points at the new stake if he takes and loses, he only stands to win 1 point at the new stake when he takes and wins, because he cashes all the games he wins — his *W* is realised in another way, the ability to cash sooner. The effective take-point (TP_O) is given by the following expression:

$$TP_O = \frac{\left(2C_V L - C_V\right)}{\left(2C_V L - C_V + 3C_V\right)} = \frac{(L-0\cdot5)}{(L+1)} \qquad \text{... (P2)}$$

When $q = TP_O$, $p = TP$ (the take-point in terms of cubeless winning probability). Therefore, from equations (P1) and (P2),

$$TP = CP \times TP_O = CP\frac{(L-0\cdot5)}{(L+1)} \qquad \text{... (P7)}$$

Consider the *single subseqent optimal redouble* model — when player A redoubles, he will be handing over a dead cube. Consequently, player A's cash-point is his dead cube cash-point, given by the following expression:

$$CP = \frac{(L + 0 \cdot 5)}{(W + L)} \qquad \text{... (P8)}$$

Substituting into equation (P7):

$$TP_1 = \frac{(L + 0 \cdot 5)(L - 0 \cdot 5)}{(W + L)(L + 1)} \qquad \text{... (P9)}$$

For other live-cube models, take-points can be derived from the following infinite series:

$$TP = tp_1(1 - tp_2(1 - tp_3(1 - tp_4(1 - tp_5(1 - tp_6(1 - tp_7(1 - \ldots\ldots \qquad \text{... (P10)}$$

where the *tp*-terms are the successive effective take-points for player A and player B respectively. If we wish to establish the take-point, in terms of cubeless winning chances for player A, then all odd-numbered *tp*-terms represent his effective take-points, and all even-numbered *tp*-terms represent player B's effective take points. The number of terms used, counting from the left, should be equal to the number of possible subsequent optimal redoubles plus one. The final term should be the dead-cube take-point for whichever side takes last.

For odd-numbered-terms, $tp_n = \frac{(L - 0 \cdot 5)}{(L + 1)}$ or $tp_n = \frac{(L - 0 \cdot 5)}{(W + L)}$ for the final-term.

For even-numbered-terms, $tp_n = \frac{(W - 0 \cdot 5)}{(W + 1)}$ or $tp_n = \frac{(W - 0 \cdot 5)}{(W + L)}$ for the final-term.

Considering, the infinite live-cube model, the take-point can be established by calculating the sum of this infinite series. By manipulation of equation (P10), the following relation may be derived:

$1 - \frac{\left(1 - \frac{TP}{tp_1}\right)}{tp_2} = TP$ which after after substitution of the relevent *tp*-terms, simplifies to:

$$TP = \frac{(L - 0 \cdot 5)}{(L + W + 0 \cdot 5)} \qquad \text{... (P6)}$$

3. Finite Model — Varying gammon and backgammon-rates

Assumptions:
1. Finite number of possible subsequent optimal redoubles.
2. Owner of cube, when he redoubles, is guaranteed to use it with perfect efficiency, at which point his opponent will have an optional pass/take.
3. Gammon and backgammon rates vary, but their rate of change is constant.

Player A and player B play backgammon for money. The win and loss values are not constant throughout the life of the game, but the rate of change of these values is — measured by change in win value per redouble, i.e., every time a redouble occurs, the win rate of the person doubled reduces or increases by constant factor. Consequently, it is no longer enough to specify only average win values (*W* and *L*). The additional parameters required for our analysis we define as follows:

y = change in win (or loss) value per redouble
w = initial average win value (immediately after the current cube action)
l = initial average win value (immediately after the current cube action)

Take-points can be derived from the following infinite series:

$$TP = tp_1(1-tp_2(1-tp_3(1-tp_4(1-tp_5(1-tp_6(1-tp_7(1-\ldots\ldots \qquad \text{... (P10)}$$

where the *tp*-terms are the successive effective take-points for player A and player B respectively. If we wish to establish the take-point, in terms of cubeless winning chances for player A, then all odd-numbered *tp*-terms represent his effective take-points, and all even-numbered *tp*-terms represent player B's effective take points. The number of terms used, counting from the left, should be equal to the number of possible subsequent optimal redoubles plus one. The final term should be the dead-cube take-point for whichever side takes last.

For odd-numbered-terms, $tp_n = \dfrac{\left(0\cdot5+(l-1)y^{n-1}\right)}{\left(2+(l-1)y^{n-1}\right)}$ or

$$tp_n = \frac{\left(0\cdot5+(l-1)y^{n-1}\right)}{\left(2+(l-1)y^{n-1}+(w-1)y^{n-2}\right)}$$ for the final-term.

For even-numbered-terms, $tp_n = \dfrac{\left(0\cdot5+(w-1)y^{n-1}\right)}{\left(2+(w-1)y^{n-1}\right)}$ or

$$tp_n = \frac{\left(0\cdot5+(w-1)y^{n-1}\right)}{\left(2+(w-1)y^{n-1}+(l-1)y^{n-2}\right)}$$ for the final-term.

Curiously, numerous trials with different y-rates, indicate that the rate of change in win values does not effect the take point, if average win and loss values are considered, i.e.,

$$TP = \frac{(L-0\cdot5)}{(L+W+0\cdot5)} \qquad \text{... (P6)}$$

A full proof of this phenomenon has not been made.

Appendix 6: Letters to Danny Kleinman

5th December 1993

Dear Danny,

Re: *Take-points in Money Games*

Thanks for your letters of November 8 and 19 regarding the article I sent you. You have raised some helpful and interesting points which I will attempt to address.

A Simpler Dead-Cube Model

I agree that, as regards the *dead-cube model*, we can deal with one variable, the *gammon-adjusted* winning probability (*R*), instead of two variables representing average sizes of wins and losses (*W* and *L*). However, it doesn't necessarily follow that this method is valid for **all** degrees of cube-life. Essentially, what we have calculated is the equivalent cubeless winning probability for a gammonless game which will result in the same cubeless equity — it would be nice if this was the same equivalent gammonless probability for different positions of the cube, but it might not be. When I began this work, I had hoped to find that this was indeed the case, but now I'm almost certain that it isn't — however, the assumption yields fairly reasonable estimates of equity. Assuming for the moment that my *dead-cube* and *live-cube* formulae are correct, we can define separate *gammon-adjusted* winning probabilities in terms of *W* and *L*, as follows:

1. *Dead-Cube*

Let p be the cubeless winning probability, R_{dead} be the *gammon-adjusted* winning probability, and E_O be the *cube-owned* equity (same here for any position of the cube, or cubeless), then,

$$E_O = p(W + L) - L = 2R_{dead} - 1$$

$$\therefore R_{dead} = p\frac{(W + L)}{2} + \frac{(1 - L)}{2} \qquad \text{..(A1)}$$

2. *Live-Cube*

Let p be the cubeless winning probability, R_{live} be the cubeless *gammon-adjusted* winning probability, and E_O be the *cube-owned* equity, then,

$$E_O = p(W + L + 0 \cdot 5) - L = 2 \cdot 5R_{live} - 1$$

$$\therefore R_{live} = p\frac{(W + L + 0 \cdot 5)}{2 \cdot 5} + \frac{(1 - L)}{2 \cdot 5} \qquad \text{...(A2)}$$

By substitution from equation (A1),

$$R_{live} = 0 \cdot 8R_{dead} + 0 \cdot 2p \qquad \text{...(A3)}$$

Inspection shows that the dead-cube and live-cube gammon-adjusted winning probabilities are only equal to one another in gammonless games.

Another way of looking at this phenomenon is to inspect the take-point formulae for dead-cube, live-cube, and the *gammon-adjusted* live approximation (gala), again in terms of W and L:

$$TP_{dead} = \frac{(L - 0 \cdot 5)}{(W + L)} \qquad TP_{live} = \frac{(L - 0 \cdot 5)}{(W + L + 0 \cdot 5)} \qquad TP_{gala} = \frac{(L - 0 \cdot 6)}{(W + L)}$$

I have derived the TP_{gala} formula to yield the same answers as the gammon-adjusted winning probability method. Note again that the live-cube take points only coincide in gammonless games ($TP = 0.2$).

I have assumed, from the contents of your letter, that you accept that my live-cube take point formula is correct. This assumption is of course crucial to the above argument. Please tell me if you require any further proof (the one in the article is only approximate).

Correction to Table 1c

Well spotted. I must have read over this section numerous times without noticing what now appears an obvious error. I think you have made a similar error, as the first term in your corrected sequence should be 35.3%.

Cubeless Take-Equity Tables

The cubeless equity, for a given position, I would define as the average expected rate of profit (ppg), when the remainder of the game is played out cubeless, at the stake of 1 point, with both gammons and backgammons counting. The cubeless take-equity (cte) is the underdog's cubeless equity at the point where he/she has an optional pass/take. Considering the straightforward case when neither player has any gammon expectation:

Where the cube is ***dead***, cte = 0.75 - 0.25 = 0.5 ppg.

Where the cube is ***live***, cte = 0.80 - 0.20 = 0.6 ppg.

The above two values represent the limits of the *take-equity envelope* for gammonless games.

For games where there is some gammon expectation, the cubeless equity for any position may be defined as follows:

$$E_{Cubeless} = pW - (1 - p)L = p(W + L) - L$$

where, p is the cubeless winning probability, W is the average value of those games ultimately won, and L is the average value of those games ultimately lost. Assuming the cubeless take-point (TP) is known, then,

$$p = TP, \quad \therefore \quad E_{take} = TP(W + L) - L \quad \text{i.e., equation (4) in the article.}$$

Turn-Points in Gammonless Games

Again, well spotted. The final sentence of my ***Other Cube Action Decisions*** section should read "Maximum divergence occurs when $\boldsymbol{x}$ is about 0.57, and typically ranges between 2.00% (W = 2, L = 2) and 3.75% (W = 1, L = 1)." — a typographical error. Interestingly, when gammon-rates are 100% for both sides, maximum divergence occurs when x is about 0.58, about 0.02 more than in the gammonless situation you accurately calculated. Consequently, I have used 0.57 as an average value over the whole range of gammon-rates.

Other Formulae

These were derived from the equity formulae (equations 5-7), and a Jacoby effects adjustment method discussed later. The cube-owned and cube-unavailable equity formulae can be readily established from the take point formula. The cube-centred equity formula is derived from the assumption that the game is *effectively* played between the players' respective *effective* cash-points (curiously these are the too-good points — cube-owned equity = the value of the cube before doubling).

Use of Variables and Formulae

I agree with the points you raise regarding how backgammon players think. However, if the formulae were to use more conventional variables, they would be unwieldy and over-complex in all but the simpler *dead* case, (where they aren't really needed anyway). Perhaps it would have been better to have constructed the *take-point* and *take-equity tables* using gammon-rates instead of W and L, but they then wouldn't have dealt with backgammons. Trying to develop similar cube-formulae using only conventional variables is extremely difficult. I adopted this approach some two years ago, without any useful result — just waste-paper, dead brain-cells, and a feeling of intellectual impotence. When it occurred to me to use average win and loss values instead, I was delighted to find the formulae penetrating themselves out of the fog of my ignorance — they were more discovered than invented.

The formulae modelling Jacoby effects I am much less comfortable with myself. Essentially, what I have done is to define what happens at the extremes of cube-life in terms of k, a contrived (rather than discovered) no-Jacoby initial double-point multiplier — we know what happens when the cube is *dead* (Kauder's and Latto's paradoxes) and *live* (no effect whatsoever). In between, with current understanding, lies that fog I mentioned before. I have used the simplest algorithm, which satisfies the known criteria at the extremes, to chart a path through what I believe is now a light mist — approximating to roughly linear interpolation. Even if the relationship is non-linear, i.e., curved, in what direction should it curve and what should its curvature be — I don't know, does anybody? Other methods and approaches are equally valid, but I doubt significantly more accurate. I think a formula has no business being refined and over-complicated without just and proven cause.

I hope I have managed to clarify some of my thinking on the interesting points you raised in your letters. It goes without saying that I would be pleased if you have any further feedback. I was especially pleased to receive your speedy and detailed response, and your valuable proof reading for that matter. I take the opportunity to enclose another article I've written, this time on the use of statistical theory to quantify the significance of rollout results. Again any feedback would be much appreciated. Best wishes for Christmas and the New Year — this is an unnecessary restriction, simply best wishes is more appropriate.

Yours sincerely,

Rick Janowski

3rd January 1994

Dear Danny,

Thanks for your letter of 22nd December — I was pleased to find that it wasn't ***hand-written***. I include an additional appendix to my article on money-game take points, which shows how the take point formulae were derived. I was pleased that you enjoyed my article on rollout statistics, but tell me, do you agree with my comments about how appropriate the *random* analysis is to backgammon simulations (pages 4 and 11 of the article)? Please pass on my thanks to Nicole for her greetings and uncancelled stamps.

Yours sincerely,

Rick Janowski

8th January 1994

Dear Danny,

Thanks for your recent letter (postmark 28th December). I like your article, and would agree that it is easier reading than my own, whilst still conveying the most salient points, along with your own special insights. I noticed two minor typographical errors, which you may have spotted already:

1. ***The Janowski Formula*** — Paragraph beginning “ The following example (position omitted) ...”, 2nd sentence: "Likkewise"

2. ***The Janowski Factor*** — Final paragraph, 4th sentence: “posit" — I don't know what should be written, but I know what you mean.

I also have some observations:

Typical J-Values

I agree that, as regards take-points, my typical J-value of 0.67 is a little high, and consider a value of about 0.60 more appropriate — similar to your estimate of $^4/_7$. As regards doubling points, however, the value of 0.67 is, I believe a good estimate — the cube-owner is closer to his target so he is less likely to miss by much. As you rightly point out, the take-point is relatively insensitive to the J-Factor assumed. This is not the case with doubling-points, which are far more sensitive. Hence, I decided to use 0.67 as the best compromise value, as by this means no substantial errors should occur — an intuitive *least squares* solution if you would. These considerations don't apply of course when the *Refined General Model* is considered (Appendix 2) — I would estimate typical values of 0.75 and 0.60, for J1 and J2 respectively.

Application of Doubling Formulae

I agree that considerations of *average* volatility are more appropriate to take-points than doubling points — the long-term volatility of the former cannot easily be assessed, whereas the short-term volatility of the latter can be. Moreover, the different degrees of sensitivity, mentioned previously, are relevant. Assessing double/no double decisions in such a general manner is not entirely useless — you have a *ballpark* figure to work with which should not normally result in significant errors. Furthermore, doubling errors are far smaller than take/pass errors in terms of probability difference from the relevant threshold. The liveness of the cube that you contemplate when turning the cube is just as average when in a take situation — granted, in the former the liveness has a more solid shape, in comparison to the haziness of the latter, but mean or average values are unaffected by the degree of understanding of the likely scenarios — average does not mean unclear. Having said all that, I believe that, where possible, it is advisable to estimate the volatility of the position, before applying the doubling formulae. It might be said that this is a waste of time, as you might as well do a detailed calculation, having regard to specific market-losers. However, it is possible to make reasonable estimates of volatility over the board, from experience, knowledge from reference positions, simplified calculations, or plain instinct. Such a reasonable estimate of volatility, combined with rollout or estimated results, would enable the formulae to provide sound advise on cube-action (particularly so using the *Refined General Model*). Similarly, computers are able to estimate volatility (or soon will be) — they can look at the resultant estimated equity swings on the 1296 combinations, providing a fairly accurate value of volatility, even if its equity estimates are less than reasonable. Long-term volatility is more difficult to assess accurately — rollouts are probably necessary, until enough reference positions are known so that family characteristics are recognisable. With regard to races, I enclose an additional appendix to my article, containing some examples which you might find interesting.

In closing, I would like to say that I thoroughly enjoyed your article — I particularly like your concept of the J-Factor being an increment of an average win. I had formulated a similar idea, but didn't explain it nearly so well — if I did at all. I feel both proud and a little embarrassed at the praise you poor upon me. We British are so modest — but what is modesty if not humility without sincerity? Enough introspective nonsense. Again, it goes without saying that I would appreciate any comments you might have. By the way, are you planning to publish a new book soon? That would be something to look forward to.

Yours sincerely,

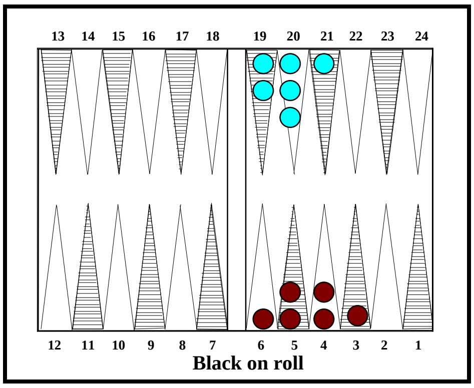
Rick Janowski

Appendix 7: Examples — Refined General Model

1. Simple Race Positions

In the following three examples, cube-life indices, x_1 and x_2,are calculated from known position equities (from Walter Trice's *Quizmaster*). The accuracy of the model is then investigated by comparing the calculated centred-cube equity with the actual centred-cube-equity.

Example 7-1a



Position: 001221-000132	**cwp** = 0.778993
Cube Position	**Equity**
Black owns 2-cube	+1.633871
White owns 2-cube	+1.042959
Centred-cube	+0.797339

From equation (8):

$$x_1 = 2\left[\frac{1}{p}\left(\frac{E_O}{C_V} + L\right) - W - L\right] = 2\left[\frac{1}{0.778993}\left(\frac{1.633871}{2} + 1\right) - 1 - 1\right] = 0.664831$$

From equation (9):

$$x_2 = 2\left[\frac{1}{(1-p)}\left(W - \frac{E_O}{C_V}\right) - W - L\right] = 2\left[\frac{1}{(1-0.778993)}\left(1 - \frac{1.042959}{2}\right) - 1 - 1\right] = 0.330365$$

From equation (10):

$$Q_x = \left(\frac{W + L + 0.5x_2}{W + L + 0.5x_1}\right) = \left(\frac{1+1+0.5\times 0.330365}{1+1+0.5\times 0.664831}\right) = 0.928301$$

and

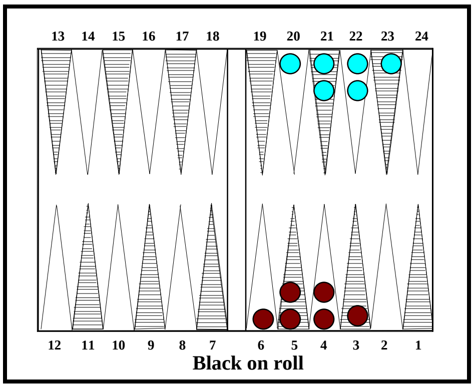
$$E_C = \frac{1\times[2\times 0.778993(1+1+0.5\times 0.330365) - 0.928301(1+1) - (1+0.5\times 0.330365 - 1)]}{[0.928301(1+1) - (1+0.5\times 0.330365 - 1)]} = 0.799056$$

Alternatively, from equation (11), a more approximate value can be established as follows:

$$E_C \approx 4\times 1\times \frac{(0.664831\times 1.633871\times 0.5 + 0.330365\times 1.042959\times 0.5)}{[4(0.664831 + 0.330365) - 2\times 0.664831\times 0.330365]} = 0.808020$$

Both the calculated values compare favourably with the true value of 0.797339.

Example 7-1b



Position: 001221-012210 **cwp** = 0.492070	
Cube Position	**Equity**
Black owns 2-cube	+0.218114
White owns 2-cube	-0.337572
Centred-cube	-0.048802

From equation (8):

$$x_1 = 2\left[\frac{1}{p}\left(\frac{E_O}{C_V}+L\right)-W-L\right] = 2\left[\frac{1}{0.492070}\left(\frac{0.218114}{2}+1\right)-1-1\right] = 0.507720$$

From equation (9):

$$x_2 = 2\left[\frac{1}{(1-p)}\left(W-\frac{E_O}{C_V}\right)-W-L\right] = 2\left[\frac{1}{(1-0.492070)}\left(1+\frac{0.337572}{2}\right)-1-1\right] = 0.602154$$

From equation (10):

$$Q_x = \left(\frac{W+L+0.5x_2}{W+L+0.5x_1}\right) = \left(\frac{1+1+0.5\times 0.602154}{1+1+0.5\times 0.507720}\right) = 1.020949$$

and

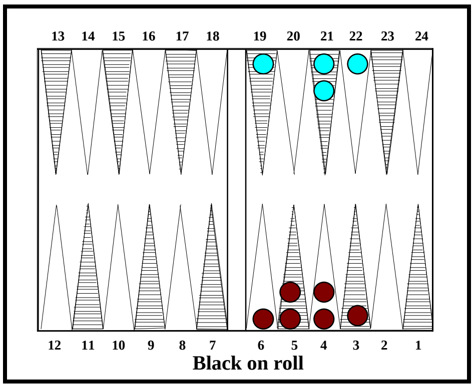
$$E_C = \frac{1\times[2\times 0.492070(1+1+0.5\times 0.602154)-1.020949(1+1)-(1+0.5\times 0.602154-1)]}{[1.020949(1+1)-(1+0.5\times 0.602154-1)]} = -0.045032$$

Alternatively, from equation (11), a more approximate value can be established as follows:

$$E_C \approx 4\times 1\times\frac{(0.507720\times 0.218114\times 0.5-0.602154\times 0.337572\times 0.5)}{[4(0.507720+0.602154)-2\times 0.507220\times 0.602154]} = -0.048335$$

Both the calculated values compare favourably with the true value of -0.048802.

Example 7-1c



Position: 001221-001201	**cwp** = 0.236591
Cube Position	**Equity**
Black owns 2-cube	-0.989290
White owns 2-cube	-1.673142
Centred-cube	-0.820455

From equation (8):

$$x_1 = 2\left[\frac{1}{p}\left(\frac{E_O}{C_V}+L\right)-W-L\right] = 2\left[\frac{1}{0.236591}\left(\frac{-0.989290}{2}+1\right)-1-1\right] = 0.271971$$

From equation (9):

$$x_2 = 2\left[\frac{1}{(1-p)}\left(W-\frac{E_O}{C_V}\right)-W-L\right] = 2\left[\frac{1}{(1-0.236591)}\left(1+\frac{1.673142}{2}\right)-1-1\right] = 0.811501$$

From equation (10):

$$Q_x = \left(\frac{W+L+0.5x_2}{W+L+0.5x_1}\right) = \left(\frac{1+1+0.5\times 0.811501}{1+1+0.5\times 0.271971}\right) = 1.126295$$

and

$$E_C = \frac{1\times[2\times 0.236591(1+1+0.5\times 0.811501) - 1.126295(1+1) - (1+0.5\times 0.811501-1)]}{[1.126295(1+1) - (1+0.5\times 0.811501-1)]} = -0.823018$$

Alternatively, from equation (11), a more approximate value can be established as follows:

$$E_C \approx 4\times 1\times \frac{(-0.271971\times 0.989290\times 0.5 - 0.811501\times 1.673142\times 0.5)}{[4(0.271971+0.811501) - 2\times 0.271971\times 0.811501]} = -0.835876$$

Both the calculated values compare favourably with the true value of -0.820455.

Summary and Discussion

Both equations (10) and (11) give good estimates of the cube-centred equity — the former particularly so, as can be seen from the summary of results tabulated below:

Example	**7.1a**	**7.1b**	**7.2c**
cwp	0.778993	0.492070	0.236591
x_1	0.664831	0.507720	0.271971
x_2	0.330365	0.602154	0.811501
$x_1 + x_2$	0.995196	1.109874	1.083472
E_C actual	+0.797339	-0.048802	-0.820455
E_C equation (10)	+0.799056	-0.045032	-0.823018
E_C equation (11)	+0.808020	-0.048335	-0.835876

Notice that although the individual values for x_1 and x_2 vary from example to example, the sum of x_1 and x_2 is fairly consistent. This suggests that both players have a shared *pool* of cube-life to draw upon — the player with the better probability takes the greater share as he is nearer to his doubling target. It would therefore appear possible to construct an algorithm for assessing the shared cube-life pool, from various pertinent factors — length of race and its standard deviation, and average bearoff wastage being the most critical. The distribution of this pool could then be assigned by another algorithm from the above mentioned factors and the cubeless probability. There are four extreme bearoff wastage conditions, which are, fortunately, readily calculable:

1. single checker versus single checker
2. no-miss position versus no-miss position
3. single checker versus no-miss position
4. no-miss position versus single checker

These four extreme conditions can be imagined as a rectangular envelope, encompassing all other intermediate conditions, whose relevant cube coefficients can be interpolated by some means. One factor the above method of analysis would not allow for is any special conditions which effect cube usage, e.g. a heavy 2-point in the final stages of a race tends to generate effective doubling positions, whilst a heavy ace-point does not. Such phenomenon are not so common in races longer than about 20 pips, but the overall effect would need to be investigated. Interestingly, if this method proves valid to money games, it could quite easily be extended to matches. This would give valuable information on how normal cube actions are modified at certain match scores and cube-levels. Moreover, a general knowledge of cube-potency at specific match scores would improve our understanding of match equities.